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國立高雄大學九十三學年度研究所碩士班招生考試試題

系所組別：統計學研究所　　　　科目：基礎數學

考卷共兩張,滿分為100分。作答時請於答案卷上註明題號及附上推導過程。

1. Suppose $h(t)$ is a nondecreasing continuous function for $t \geq 0$. Let $H(t) = \int_0^t h(x)dx$. Is $\frac{H(t)}{t}$ nondecreasing for $t > 0$? Justify your answer. (10%)

2. Which is larger (for $n > 8$): $(\sqrt{n})^{\sqrt{n+1}}$ or $(\sqrt{n+1})^{\sqrt{n}}$? (12%)

3. (a) Evaluate $\lim\limits_{x \to 3} \dfrac{x}{x-3} \displaystyle\int_3^x \frac{\sin t}{t} dt$. (6%)

   (b) Evaluate $\lim\limits_{n \to \infty} \displaystyle\int_0^1 \frac{ny^{n-1}}{1+y} dy$. (10%)

4. Define a sequence by
$$a_n = \int_0^1 (1-x^2)^n \, dx.$$
   Show that $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$. Does $\sum_{n=1}^{\infty} a_n$ converge? (12%)

5. Let $M_{n\times n}(\mathbf{R})$ be the set of all $n \times n$ matrices over $\mathbf{R}$.

   (a) Show that $\{I, A, A^2, \ldots, A^n\}$ is linearly dependent for all $A \in M_{n\times n}(\mathbf{R})$. (5 %)

   (b) Let $A \in M_{n\times n}(\mathbf{R})$. Show that A is invertible if and only if I belongs to the span of $\{A, A^2, \ldots, A^n\}$. (7 %)

6. Let A be an $n \times n$ real skew-symmetric matrix; i.e., $A^t = -A$, where A^t denotes the transpose of A.

   (a) Show that $\det(A) - (-1)^n \det(A) = 0$. (4 %)

   (b) If n is odd, show that A must be singular. (3 %)

   (c) What can you say about the eigenvalues of A? (6 %)

7. Let $T : M_{n\times n}(\mathbf{F}) \longrightarrow M_{n\times n}(\mathbf{F})$ be the linear transformation defined by $T(A) = \dfrac{A + A^t}{2}$, where $M_{n\times n}(\mathbf{F})$ denotes the set of all $n \times n$ matrices over a field $\mathbf{F}$.

   (a) What is Ker(T) (kernel of T)? (6 %)

   (b) What is the dimension of Ker(T)? (7 %)

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8. Find all possible 2×2 real matrices A such that

$$A^4 = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}. \ (12\%)$$

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系所組別：統計學研究所　　　　科目：數理統計

每題20分, 該有的步驟須附上。

1. 設$X_1,\cdots,X_n$為一組由 $\mathcal{U}(0,\theta)$ 分佈所產生之隨機樣本, $\theta>0$。令$X_{(1)}=\min X_i$, $X_{(n)}=\max X_i$。

   (i) 試證 $X_{(1)}/X_{(n)}$ 與 $X_{(n)}$ 獨立。(5分)

   (ii) 指出 $X_{(1)}/X_{(n)}$ 為那一常見分佈, 並給出參數。(5分)

   (iii) 指出 $X_{(n)}/\theta$ 為那一常見分佈, 並給出參數。(5分)

   (iv) 試求 $X_{(1)}/X_{(n)}$ 與 $X_{(n)}^2$ 之相關係數。(5分)

2. 設$X_1,\cdots,X_n$為一組由 $N(\theta,a\theta^2)$分佈所產生之隨機樣本, 其中$a>0$為一常數, $\theta>0$為未知參數。令$\overline{X}_n=\sum_{i=1}^n X_i/n$, $S_n^2=\sum_{i=1}^n(X_i-\overline{X}_n)^2/(n-1)$。

   (i) 試求$E(S_n^2)$。(5分)

   (ii) 試證$T=(\overline{X}_n,S_n^2)$ 為θ之一充分統計量, 但並非完備統計量。(15分)

3. 設$X_1,\cdots,X_n$為一組由在區域S上均勻分佈所產生之隨機樣本, 其中$S=[-2,-1]\cup[0,\theta],\theta\geq 0$.

   (i) 試求θ 之最大概似估計量(Maximum Likelihood Estimator, MLE)。(10分)

   (ii) 若只觀測到負的$X_i's$之個數U (而不知$X_1,\cdots,X_n$之值), 試求此時θ之MLE。(10分)

4. 設$X_1,\cdots,X_4$為一組由 $\mathcal{U}(0,\theta)$ 分佈所產生之隨機樣本, $\theta>0$。欲檢定$H_0:\theta=1$, v.s. $H_a:\theta\neq 1$. 令$X_{(4)}=\max\{X_1,X_2,X_3,X_4\}$. 取拒絕域為$\{X_{(4)}<1/2$, 或$X_{(4)}>1\}$.

   (i) 試求檢力函數(power function) $K(\theta),\theta>0$。(16分)

   (ii) 給出$K(1/2)$ 及$K(1)$。(4分)

5. 設$X_1,\cdots,X_n$為一組由 $\mathcal{E}(\lambda)$ 分佈所產生之隨機樣本, $\lambda>0$。令 $q(\lambda)=1-e^{-\lambda a}$, 其中$a>0$ 為一定值。

   (i) 試求 $q(\lambda)$之MLE。(5分)

   (ii) 試求 $q(\lambda)$之一致最小變異不偏估計量(Uniformly Minimum Variance Unbiased Estimator, UMVUE)。(15分)

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1. Assume X has a normal distribution with mean zero and variance σ^2. Show that $E(X^4) = 3(E(X^2))^2$. (10%)

2. Independent trials, each of which is a success with probability p, are performed until there are k consecutive successes. Let N_k denote the number of necessary trials. Find the expectation of N_k, i.e. $E(N_k)$. (10%)

3. Consider an autoregressive model,

$$X_n = -X_{n-1} + \varepsilon_n, \text{ for } n = 1, 2, \cdots,$$

where $\varepsilon_i, i = 1, 2, \cdots$, are i.i.d. random variables with $E(\varepsilon_i) = \mu$ and $Var(\varepsilon_i) = \sigma^2$, and $X_0 = 0$. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Show that $\sqrt{n}(\bar{X}_n - \mu/2)$ converges in distribution to a normal distribution with mean zero and variance $\sigma^2/2$ as $n \to \infty$. (10%)

4. Let X be a Poisson distribution with a parameter λ. Show that $(X - \lambda)/\sqrt{\lambda}$ converges in distribution to a normal distribution with mean zero and variance one as $\lambda \to \infty$. (10%)

5. Let $X_1, X_2, \cdots$, be a sequence of random variables such that X_1 is uniform$[0,1]$ and where for $n = 1, 2, \cdots$, the conditional distribution of X_{n+1} given $X_1, \cdots, X_n$ is uniform$[0, cX_n]$ for some number c such that $\sqrt{3} < c < 2$. Find the expectation of X_n^r for $r > 0$. Then show that $E(X_n) \to 0$ as $n \to \infty$ but $E(X_n^2) \not\to 0$ as $n \to \infty$. (20%)

6. **(a)** Let X be a random variable, and $g(x)$ is a convex function. Prove the following inequality (Jensen's Inequality)

$$E(g(X)) \geq g(E(X)). \text{ (10\%)}$$

(b) If $y_1, \cdots, y_n$ are positive numbers, define

$$y_A = \frac{1}{n}(y_1 + \cdots + y_n), y_G = [y_1 y_2 \cdots y_n]^{1/n}, y_H = \frac{1}{\frac{1}{n}(\frac{1}{y_1} + \cdots + \frac{1}{y_n})}.$$

Use Jensen's inequality to prove that $y_H \leq y_G \leq y_A$. (15%)

7. Assume $X_1, \cdots, X_n$ are i.i.d. uniform$(0, a)$, $a > 0$. Let $X_{(1)}, \cdots, X_{(n)}$ denote the order statistics of $X_1, \cdots, X_n$. Define $R = X_{(n)} - X_{(1)}$ and $V = (X_{(1)} + X_{(n)})/2$. Find the joint distribution of (R, V) and the marginal distributions of R and V. (15%)