國立高雄大學九十四學年度研究所碩士班招生考試試題

系所組別：統計學研究所　　　　科目：基礎數學

考試時間：100 分鐘　　　　本科原始成績滿分 100 分

1. (10 points) Show that $(1+x)^{1/x} = e\cdot(1-\frac{x}{2}+\frac{11x^2}{24}+o(x^2))$ as $x\to 0$.

2. (10 points) Prove that $\lim_{x\to 0}(1+ax)^{1/x} = e^a$ for every real a.

3. (16 points) Let $a_{2n-1} = \frac{1}{n}$ and $a_{2n} = \int_n^{n+1}\frac{dx}{x}$ for $n = 1, 2, 3, \cdots$.

   (a) Show the series $s_n = \sum_{i=1}^{n}(-1)^{i-1}a_i$ converges. In fact, s_n converges to *Euler's constant.*

   (b) Use the result of (a) to show $\sum_{k=1}^{\infty}(-1)^{k-1}/k = \log 2$.

4. (12 points) Evaluate

   (a) $\int_0^{\pi/2}\sin^2 x dx$

   (b) $\int_0^{\pi/2}\sin^4 x dx$

5. (12 points) Consider the $p\times p$ matrix $X'X$, and X' is the transpose of X. Let x' be the ith row of X. Hence $X'X - xx'$ is the $X'X$ matrix with the ith row removed. Assume $(X'X)^{-1}$ exists. Show that

$$(X'X-xx')^{-1} = (X'X)^{-1} + \frac{(X'X)^{-1}xx'(X'X)^{-1}}{1-x'(X'X)^{-1}x}.$$

6. (10 points) Define the 2×2 matrix, $A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$. Find a 2×2 matrix B such that $A = BB'$.

7. (20 points) Let the 4×4 matrix C be defined by

$$C = \begin{pmatrix} 2 & 4 & 4 & 4 \\ 4 & 2 & 4 & 4 \\ 4 & 4 & 2 & 4 \\ 4 & 4 & 4 & 2 \end{pmatrix}.$$

   Find the determinant of C and the inverse matrix of C.

8. (10 points) Let A be a $p\times p$ idempotent matrix and let B be a $p\times p$ tripotent matrix. That is

$$A = A^2 \text{ and } B = B^3.$$

   Find the all possible eigenvalues of A and B.

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考試時間：100分鐘　　　　　　　　　　　本科原始成績滿分 100分

試卷共有兩張。作答時請於答案卷上註明題號及附上推導過程。

1. Let X_1, X_2 and X_3 be independent random variables with mean δ, 2δ and 3δ, respectively, and the same variance σ^2. The following three estimates are all unbiased for δ:

$$\hat{\delta}_1 = \frac{1}{3}\left(X_1 + \frac{1}{2}X_2 + \frac{1}{3}X_3\right), \ \hat{\delta}_2 = \frac{1}{6}(X_1 + X_2 + X_3), \ \hat{\delta}_3 = \frac{1}{14}(X_1 + 2X_2 + 3X_3).$$

   (a) Which of these three estimators has the smallest variance? (6 %)

   (b) Is there a linear unbiased estimator of δ with smaller variance than any of the three suggested above? Justify your answer. (10 %)

2. Let $X_1, \ldots, X_n$ be a random sample from the density

$$f(x;\theta) = \begin{cases} e^{-(x-\theta)}, & \theta \le x < \infty, \\ 0, & \text{otherwise}, \end{cases}$$

   where $-\infty < \theta < \infty$.

   (a) Find the MLE (maximum-likelihood estimator) of θ. (8 %)

   (b) Find an estimator of θ by the method of moments and show that it is consistent. (12 %)

3. Let $X_1, \ldots, X_n$ be a random sample from the Poisson density

$$f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!}, \ x = 0, 1, \ldots,$$

   where $\theta > 0$.

   (a) Show that $S = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic. (5 %)

   (b) Let $Y = (-1)^{X_1}$. Show that Y is unbiased for $e^{-2\theta}$ and conclude that it is an unreasonable estimator. (5 %)

   (c) Based on the results of (a) and (b), find an UMVUE (uniformly minimum-variance unbiased estimator) for $e^{-2\theta}$. (12 %)

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4. Let $Y_1 < Y_2 < Y_3 < Y_4$ denote the order statistics of a random sample of size 4 from the density

$$f(x;\theta) = \begin{cases} \dfrac{3x^2}{\theta^3}, & 0 < x < \theta, \\ 0 \ , & \text{otherwise.} \end{cases}$$

(a) Find $P(k\theta < Y_4 < \theta)$, where $0 < k < 1$. (10 %)

(b) Find a 95% confidence interval for θ. (6 %)

5. Let X be a single observation from the density

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1, \\ 0 \ , & \text{otherwise,} \end{cases}$$

where $\theta > 0$.

(a) In testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, find the power function and size of the test given by the following: Reject H_0 if and only if $X \geq \frac{1}{2}$. (8 %)

(b) Find a most powerful size-α test of $H_0 : \theta = 2$ versus $H_1 : \theta = 1$. (8 %)

(c) Among all possible (simple) likelihood-ratio tests of $H_0 : \theta = 2$ versus $H_1 : \theta = 1$, find that test that minimizes $\alpha + \beta$, where α and β are the respective sizes of the Type I and Type II errors. (10 %)

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1. Suppose U_1 and U_2 are two independent uniform(0,1). Let $R = \sqrt{-2\log U_1}$ and $\theta = 2\pi U_2$. Show that $X = R\cos\theta$ and $Y = R\sin\theta$ are two independent normal(0,1). (15%)

2. Let $X_1, \cdots, X_n$ be independent Poisson random variables with common mean λ. Find the conditional distribution of X_1, given $\sum_{i=1}^{n} X_i$. (15%)

3. Let X_n be a chi-square distribution with degree of freedom n. Show that $(X_n - n)/\sqrt{2n}$ converges to the stadard normal in distribution as n tend to infinity. (15%)

4. Suppose Z_1, Z_2 be two independent random variables with common distribution $f(x) = \lambda\exp\{-\lambda x\}$. Let $X = Z_1$, $Y = Z_1Z_2 + Z_2$. Find (a) $E(Y|X = x)$, (b) $E(E(Y|X))$, (c) $Var(E(Y|X))$ (d) $Var(Y|X = x)$ and (e) $E(Var(Y|X))$. (15%)

5. Let $X_1, ..., X_n$ be a random sample drawn from a population with finite variance σ^2 and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Prove that $E(S) \leq \sigma$, and if $\sigma^2 > 0$, then $E(S) < \sigma$. Does S converges to σ in probability? Why? (20%)

6. Let $X_1, \cdots, X_n$ be independent exponential random variable with common mean λ^{-1}. Denote the first r $(r < n)$ order statistics of X_i as $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$. Define $X_{(0)} = 0$. Show that $W_j = (n - j + 1)(X_{(j)} - X_{(j-1)})$, $j = 1, \cdots, r$ are also independent exponential random variable with common mean λ^{-1}. Let $T = (n - r)X_{(r)} + \sum_{i=1}^{r} X_{(i)}$. What is the distribution function of T? (Hint: Find the relation between T and W_j) (20%)