國立高雄大學九十五學年度研究所碩士班招生考試試題

科目：線性代數(甲)　　系所：應用數學系碩士班甲組　　■可 □否 使用計算機
考試時間：100 分鐘　　本科原始成績：滿分 100 分

Notations:

$M_{m\times n}(R)$: set of $m\times n$ real matrices.

$W^{\perp}$: the orthogonal complement of W.

A^T: the transpose of the matrix A.

1. (10) Find an orthonormal basis for $W=\{(x,y,z)\in R^3 \mid x+2y+z=0\}$.
2. (10) Let $W=\{(a,-a,a)\mid a\in R\}$. Find a matrix A such that $L_A: R^3\to R^3$ (defined by $L_A(x)=Ax$) is the projection on W along $W^{\perp}$.
3. Let $Q\in M_{3\times 3}(R)$ be an invertible matrix and let

$$S=\{A\in M_{3\times 3}(R)\mid AB=BA\} \quad \text{where} \quad B=Q^{-1}\begin{bmatrix}2&0&0\\0&1&0\\0&0&2\end{bmatrix}Q.$$

   a. (5) Show that S is a subspace of $M_{3\times 3}(R)$.
   b. (10) Find a basis for S.
4. a. (10) Let $A\in M_{m\times n}(R)$, $B\in M_{n\times m}(R)$. Show that if λ is a nonzero eigenvalue of AB then λ is an eigenvalue of BA.
   b. (5) Let $A=(1,\ 2,\ 3,\ 4,\ -10)^T$ and $B=(1,\ 1,\ 1,\ 1,\ 1)$. Find all the eigenvalues of AB.
   c. (5) Find the Jordan canonical form of AB.
   d. (5) Find the characteristic polynomial and minimal polynomial of AB.
5. (40) Determine each following statement either is true or false. If true, prove it; if false, give a counterexample.
   a. If $A=A^T\in M_{n\times n}(R)$ then every eigenvalue of A is real.
   b. If $A,B\in M_{n\times n}(R)$ then rank(AB)=rank(BA).
   c. Let $p(t)$ be a polynomial. If A is diagonalizable then $p(A)$ is diagonalizable.
   d. If $A\in M_{n\times n}(R)$ then rank(A^TA)=rank(A).
   e. Suppose $A,B\in M_{n\times n}(R)$. If A and B have the same characteristic polynomial and minimal polynomial then they have the same Jordan canonical form.

國立高雄大學九十五學年度研究所碩士班招生考試試題

科目：高等微積分　　系所：應用數學系碩士班甲組　　■可 □否 使用計算機

考試時間：100 分鐘　　本科原始成績：滿分 100 分

1. A set K is compact if for every open cover (G_α), there exists a finite subcover of $G_1, G_2, \cdots, G_n$ such that $K \subset \bigcup_{\alpha=1}^{n} G_\alpha$.

   (a) Find an open cover for the set $B = \{(x, y) : -1 \le x \le 1\}$ and subsequently deduce from the definition above that B is not compact in R^2. (10%)

   (b) Let $f : M \to N$ and f is continuous. If $K \subset M$ is compact then $f(K)$ is also compact. Verify the statement with $f(x, y) = 2x + y^2$ and $K = \{ (x, y) \mid x^2 + y^2 \le 1 \}$. (10%)

   (c) Is the converse of the statement in (*b*) true? Give a counter example if this is not the case. (5%)

2. Given $A \subset M$, a metric space, $x \in M$ is an accumulation point of A if every open neighbourhood of x contains a point other than x in A.

   (a) Find the accumulation points of the set $A = \{ x \mid x \in [a, b] \subset R \}$ for the cases when (i) x is real (ii) x is rational (iii) x is a natural number. (9%)

   (b) Define the condition at which a sequence is convergent. If $x \in R^n$ is an accumulation point of A show that there is a sequence (x_n) in A with $x_n \to x$. (13%)

3. (a) Given two continuous functions f and g, we can conclude that the product of the two functions $f \cdot g$ is continuous. Is this still true that $f \cdot g$ is uniformly continuous if f and g are both uniformly continuous? Give a counter example if this is not the case. (5%)

   (b) Is $f(x) = 1/x$ uniformly continuous on the set $A = (0, 1)$? If not, can you modify A and prove that $f(x)$ is uniformly continuous on the modified set A. (10%)

   (c) Is a bounded and continuous function always uniformly continuous? Give a counter example if this is not the case and justify your answer. (8%)

4. (a) Let f and g be two real functions and differentiable on $[a, b]$ in R. Assuming that $f(x_0) = 0 = g(x_0)$ and $g'(x_0) \neq 0$ for some $x_0 \in (a, b)$, prove l'Hopital's rule using the Mean Value Theorem for $f(x)/g(x)$ in the limit $x \to x_0$. (10%).

   (b) The directional derivative of f in the direction of $\mathbf{n}$ is given by $Df(x) \cdot \mathbf{n}$. Find the unit vector at $(1, 2, 1)$ in the direction that $f(x, y, z) = x^2 + y^2 + z$ increases the fastest? What is the tangent plane to the surface $f(x, y, z) = 6$ at $(1, 2, 1)$? (10%)

   (c) What is the necessary condition for a function $f : R^2 \to R$ to have Taylor's expansion? Find the second order Taylor's expansion of the function $f(x, y) = \sin x \cos y$ about the point $(\pi/2, 0)$. (10%)

國立高雄大學九十五學年度研究所碩士班招生考試試題

科目：線性代數(乙) 系所：應用數學系碩士班乙組 ■可 □否 使用計算機
考試時間：100 分鐘 本科原始成績：滿分 100 分

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2. (10) Let $W=\{(a,-a,a)\mid a\in R\}$. Find a matrix A such that $L_A : R^3\to R^3$ (defined by $L_A(x)=Ax$) is the projection on W along $W^{\perp}$.

3. (10) Let $A=\begin{bmatrix}1&1\\2&1\\3&1\\4&1\end{bmatrix}$ and $b=\begin{bmatrix}2\\3\\5\\7\end{bmatrix}$. Find a vector x that minimize $\|Ax-b\|_2$.

4. (10) Factor matrix $\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}=LL^T$ where L is a lower triangular matrix.

5. (10) Solve $\frac{du}{dt}=\begin{bmatrix}3&-1\\-1&3\end{bmatrix}u$ starting from $u(0)=\begin{bmatrix}2\\1\end{bmatrix}$ at $t=0$.

6. (10) Find the necessary and sufficient condition on c such that $\begin{bmatrix}c&4&4\\4&c&4\\4&4&c\end{bmatrix}$ is symmetric positive definite.

7. (40) Determine each following statement either is true or false. If true, prove it; if false, give a counterexample.
   a. If $A=A^T\in M_{n\times n}(R)$ then every eigenvalue of A is real.
   b. If $A\in M_{n\times n}(R)$ is similar to $-A$ then $A=0$.
   c. Let $p(t)$ be a polynomial. If A is diagonalizable then $p(A)$ is diagonalizable.
   d. A symmetric matrix can't be similar to a nonsymmetric matrix.
   e. Suppose $A,B\in M_{n\times n}(R)$. If rank(A) =rank(B) then rank(A^2) =rank(B^2).

國立高雄大學九十五學年度研究所碩士班招生考試試題

科目：微分方程　　系所：應用數學系碩士班乙組　　■可 □否 使用計算機
考試時間：100 分鐘　　本科原始成績：滿分 100 分

1. (15%) Show that the differential equation $y\cos x + 2xe^{y} + \left(\sin x + x^{2}e^{y} - 1\right)y' = 0$ is exact and then find its solution.
2. (15%) Find the general solution of $y'' - 3y' - 4y = 2\sin x$.
3. (15%) Use the Laplace transform to solve the differential equation $y'' + y = \sin 2x$ satisfying the initial conditions $y(0) = 2$ and $y'(0) = 1$.
4. (15%) Find the general solution of $y'' + y = \sec x$ by using the method of variation of parameters.
5. (20%) Consider the homogeneous system of linear differential equations $Y' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} Y$ where $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ (a) Find the general solution of this system. (b) Plot the solution $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ as a single curve (called a trajectory) in the $y_1 y_2$-plane (called the phase plane of the system) with parameter t.
6. (20%) Consider the periodic Sturm-Liouville problem: $y'' + \lambda y = 0$, $y(\pi) = y(-\pi)$, $y'(\pi) = y'(-\pi)$. (a) Find eigenvalues and eigenfunctions of this problem. (b) Choose any arbitrary 3 eigenfunctions from Part (a) and show that any two of them are orthogonal on the interval $-\pi \le x \le \pi$.