

**Notations:**

$M_{m \times n}(\mathbb{R})$ : the set of  $m \times n$  matrices with entries in  $\mathbb{R}$ .

$A^{-1}$ : the inverse of the matrix  $A$ .

$\det(A)$ : the determinant of matrix  $A$ .

$\text{rank}(A)$ : the rank of the matrix  $A$ .

1 (30) Determine each follow statement either is true or false. If true, prove it; if false, give a counterexample.

a Let  $A \in M_{n \times n}(\mathbb{R})$ , if  $A^4$  is a zero matrix, then  $\det(A) = 0$ .

b Let  $A$  and  $B$  be diagonalizable matrices, then  $A + B$  is also diagonalizable.

c If  $A, B \in M_{n \times n}(\mathbb{R})$  are invertible matrices, then  $AB$  is invertible.

2 (10) Let  $W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid a_1 + a_2 + a_3 = 0\}$ . Find a basis of  $W$  over  $\mathbb{R}$ .

3 (20) Let  $A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$  and  $b = \begin{bmatrix} 7 \\ 3 \\ 4 \\ 1 \end{bmatrix}$ .

a. (5) Find the rank of  $A$ .

b. (5) Find the solution set of  $Ax = 0$ .

c. (5) Find the solution set of  $Ax = b$ .

d. (5) Find a  $4 \times 4$  matrix  $M$  with  $\text{rank}(M) = 4 - \text{rank}(A)$  such that  $AM = 0$ , where  $0$  is the  $4 \times 4$  zero matrix.

4 (15) Let  $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 3 & 2 & 0.5 & 0 \\ 1 & 2 & 0 & -3 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 4 & 1 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

a. (5) Find the  $B^{-1}$ .

b. (5) Evaluate  $\det(B^{-1}AB)$ .

c. (5) Evaluate  $\det\left(\begin{bmatrix} AB & A^{-1}BA \\ 0 & BA^{-1} \end{bmatrix}\right)$ , where  $0$  is the  $4 \times 4$  zero matrix.

5 (25) Let  $A = \begin{bmatrix} 3 & -2 & -2 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

a. (10) Find the eigenvalues and corresponding eigenvectors of  $A$ .

b. (5) Compute the matrix  $B = A^{100} + A^2 + I$ .

c. (5) Find the eigenvalues and corresponding eigenvectors of  $B$ .

d. (5) Find a polynomial  $f(t)$  with degree 3, such that  $f(A) = 0$ .

科目：微積分

系所：應用數學系二年級

 可

使用計算機

考試時間：90 分鐘

本科原始成績：滿分 100 分

 否

You need to show all your work.

1. (10 pts) Find  $(f^{-1})'(-2)$  if  $f(x) = x\sqrt{3+x^2}$ .
2. (10 pts) Calculate  $\frac{d}{dx} \left( \int_{1/x}^{1/\sqrt{x}} \sin(t^2) dt \right)$
3. (10 pts) Find  $f^{(9)}(0)$  for  $f(x) = x \cos(x^2)$ .
4. (10 pts) Evaluate  $\lim_{x \rightarrow \infty} \left( \cos \frac{1}{x} \right)^x$
5. (10 pts) Evaluate  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 y^4}{(x^2 + y^4)^3}$
6. (10 pts) Determine whether  $\sum_{k=1}^{\infty} \frac{k^{k/2}}{k!}$  converges or diverges.
7. Find or evaluate the integral.
  - (a) (10 pts)  $\int \frac{x}{\sqrt{6x-x^2}} dx$
  - (b) (10 pts)  $\int \frac{1}{1+\cos x - \sin x} dx$
  - (c) (10 pts)  $\int_1^4 \frac{dx}{x^2-4}$
  - (d) (10 pts)  $\int_0^1 \int_{\tan^{-1} y}^{\pi/4} \sec x dx dy$