國立高雄大學九十六學年度研究所碩士班招生考試試題

科目：高等微積分　　　　系所： 應用數學系甲組　　　　是否使用計算機：是
考試時間：100 分鐘　　　本科原始成績：100 分

1. Given a sequence of continuous functions $f_k : A \subset M \to N$, M and N being two metric spaces.

   (a) If $f_k \to f$, is f continuous? Give a counter example if this is not true. (6%)

   (b) Prove that if $f_k \to f$ uniformly on A then f is continuous on A. (13%)

   (c) Find the limit function of the sequence $f_k(x) = x^k - x - 1$ for $0 \leq x \leq 1$ and determine the convergence of the sequence. (6%)

2. Given a function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $x_0 \in A$.

   (a) Define the condition at which f is differentiable at x_0 and subsequently show that $f(x,y) = (xy, y)$ is differentiable at $(x_0, y_0) \in A \subset \mathbb{R}^2$. (10%)

   (b) Alternatively, f is said to be differentiable at x_0 if given $\epsilon > 0$, $\exists\, \delta > 0$, such that for all $\|x - x_0\| < \delta$, $\|f(x) - f(x_0) - Df(x_0)(x - x_0)\| < \epsilon\|x - x_0\|$. Use this statement to show that if $f(x)$ is differentiable then $x^2 f(x)$ is also differentiable at x_0 and the derivative is given by $D(x_0^2 f(x_0)) = 2x_0 f(x_0) + x_0^2 Df(x_0)$. (15%)

3. Given $f : A \to \mathbb{R}^n$. Let B be a rectangle containing A, i.e. $A \subset B$ and $\mathcal{P}(B)$ is the set of all possible partitions of B. Let $U(f,P)$ and $L(f,P)$ denote the upper and lower sum of f and define $s = \sup\{L(f,P) | P \in \mathcal{P}(B)\}$ and $S = \inf\{U(f,P) | P \in \mathcal{P}(B)\}$.

   (a) Define the condition at which f is Riemann integrable and subsequently use the definition to evaluate the integral $\int_0^3 (x+5)\,\mathrm{d}x$. (10%)

   (b) Alternatively, f is said to be integrable iff for any $\epsilon > 0$ there exists a partition P_ϵ of B such that $0 \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$. Use this statement to show that if f is continuous and $A = [a,b]$ then f is integrable. (15%)

4. Let $f : A \subset \mathbb{R}^3 \to \mathbb{R}$ be differentiable on A and $f(x,y,z) = x^2 + y^2 + z^2 + xy + yz + zx$.

   (a) Find the critical point of f and determine the nature of it. (10%)

   (b) Find the critical point of f subject to the constraint $g(x,y,z) = x + y + z$ with $\{(x,y,z) \in \mathbb{R}^3 \,|\, g(x,y,z) = 1\}$. (5%)

   (c) Let S be the surface $f(x,y,z) = 2$. Find the unit normal to the surface S at $c = (1,-1,1)$ and subsequently determine the tangent plane to the surface at c. (5%)

   (d) What is the directional derivative of f at c in the direction that the rate of change is the greatest ? (5%)

國立高雄大學九十六學年度研究所碩士班招生考試試題

科目：線性代數(甲)　　系所： 應用數學系甲組　　是否使用計算機：是
考試時間：100 分鐘　　本科原始成績：100 分

Notations.

I_n : the identity matrix of size n.
$[T]_\beta^\gamma$: matrix representation of T relative to ordered bases β, γ.
$M_{n\times m}(\mathbb{R})$: set of $n \times m$ real matrices.
$A^\top$: the transpose of matrix A.

1 Let $H = \text{span}\{(1,-1,0,2)^\top, (2,1,-2,0)^\top, (0,-3,2,4)^\top, (3,3,-4,-2)^\top\}$.
a. (6) Find a basis for H.
b. (10) Which vector in H is closed to $(7,5,0,3)^\top$.

2 (10) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear and let $\beta = \{(1,1),(0,1)\}$ $\gamma = \{(1,1,0),(0,1,1),(1,0,1)\}$ be ordered bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. If

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}$$

then find a matrix A such that $T(x) = Ax$.

3 Let $\mathbf{v}$ be a unit vector in $\mathbb{R}^n$ and $A = I_n - 2\mathbf{v}\mathbf{v}^\top$.
a. (6) Show that $\|A\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^n$.
b. (6) Find the eigenvalues and corresponding to eigenspaces of A.
c. (10) Let $\mathbf{x} = (1,1,0)^\top$. Find a vector $\mathbf{v} \in \mathbb{R}^3$ such that $A\mathbf{x} = (\sqrt{2},0,0)^\top$.

4 Given

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

a. (10) Find the Jordan canonical form J of A and a matrix Q such that $Q^{-1}AQ = J$.
b. (3) Find the minimal polynomial of A.
c. (5) Find scalars $a, b, c \in \mathbb{R}$ such that $A^{-1} = aI_3 + bA + cA^2$.

5 (10) Suppose $A = A^\top \in M_{n\times n}(\mathbb{R})$. If λ_1 and λ_2 are distinct eigenvalues of A with corresponding eigenvectors x_1 and x_2 then show that x_1 and x_2 are orthogonal.

6 (24) Determine "true" or "false" for the following statements. Briefly sketch your proof when the answer is "true", or give a counterexample when the answer is "false".

a The intersection of two subspaces of a vector space is a subspace.

b If $A \in M_{4\times 4}(\mathbb{R})$ and $A^2 + 3A + 2I_4 = 0$ where 0 is a zero matrix, then A is invertible.

c If $A \in M_{3\times 3}(\mathbb{R})$ and $\text{rank}(A) = 1$ then A has a nonzero eigenvalue.

d Let $A,\ B \in M_{n\times n}(\mathbb{R})$. The determinant of $AB - BA$ is zero.

國立高雄大學九十六學年度研究所碩士班招生考試試題

科目：微分方程　　系所： 應用數學系乙組　　是否使用計算機：是
考試時間：100 分鐘　　本科原始成績：100 分

1. (30%) Find the general solution of the following differential equations:
   (a) $e^y dx + (xe^y + 2y)\, dy = 0$.
   (b) $ydx + \left(x^2 y - x\right) dy = 0$.
   (c) $y'' + y = \sin x$.

2. (15%) Verify the $y_1(x) = x$ and $y_2(x) = 1/x$ are solutions of
   $$x^2 y'' + xy' - y = 0,$$
   and then find the general solution of
   $$x^2 y'' + xy' - y = x \ln x.$$

3. (20%) Consider the homogeneous system of linear differential equations $X'(x) = AX(t)$, where
   $$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$
   (a) Find the eigenvalues and eigenvectors of A.
   (b) Find the general solution of the system.
   (c) Classify the critical point $(0, 0)$ as to type and determine whether it is stable, asymptotically stable, or unstable.

4. (20%) Consider the Sturm–Liouville problem
   $$y'' + \lambda r(x)\, y = 0,\ x \in (0, 1)$$
   with the boundary condition
   $$a_1 y(0) + a_2 y'(0) = 0 \text{ and } b_1 y(1) + b_2 y'(1) = 0,$$
   where $r(x)$ is a positive continuous function on the interval $[0, 1]$ and $a_1, a_2, b_1, b_2 \in \mathbb{R}$.
   (a) Show that all eigenvalues λ are real.
   (b) Suppose that ϕ_1 and ϕ_2 are two eigenfunctions of problem corresponding to eigenvalues λ_1 and λ_2, respectively. Show that if $\lambda_1 \neq \lambda_2$, then
   $$\int_0^1 r(x)\, \phi_1(x)\, \phi_2(x)\, dx = 0.$$

5. (15%) Use the Laplace transform to solve the differential equation
   $$y'' + 4y = 4x$$
   that satisfies the initial conditions $y(0) = 1$ and $y'(0) = 5$.

國立高雄大學九十六學年度研究所碩士班招生考試試題

科目：線性代數(乙)　　系所： 應用數學系乙組　　是否使用計算機：是
考試時間：100 分鐘　　本科原始成績：100 分

Notations.

I_n : the identity matrix of size n.
$M_{n\times m}(\mathbb{R})$: set of $n\times m$ real matrices.
$A^\top$: the transpose of matrix A.

1 Let

$$A=\begin{bmatrix} 1 & 2 & 1\\ 2 & 5 & 4\\ -2 & 0 & 6\\ 1 & 4 & 5\end{bmatrix}.$$

a. (15) Find a basis for the row space, a basis for the column space and a basis for the null space of the matrix.
b. (8) Which vector in column space of A is closed to $(4,2,6,6)^\top$.

2 Let $\mathbf{v}$ be a unit vector in $\mathbb{R}^n$ and $A=I_n-2\mathbf{v}\mathbf{v}^\top$.
a. (6) Show that $\|A\mathbf{x}\|_2=\|\mathbf{x}\|_2$ for all $\mathbf{x}\in\mathbb{R}^n$.
b. (6) Find the eigenvalues and corresponding to eigenspaces of A.
c. (10) Let $\mathbf{x}=(1,1,0)^\top$. Find a vector $\mathbf{v}\in\mathbb{R}^3$ such that $A\mathbf{x}=(\sqrt{2},0,0)^\top$.

3 Given

$$A=\begin{bmatrix} 3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{bmatrix}$$

a. (8) Find the eigenvalues and corresponding eigenvectors of A.
b. (3) Find the minimal polynomial of A.
c. (5) Find scalars $a,b\in\mathbb{R}$ such that $A^{-1}=aI_3+bA$.
d. (5) Find the minimum of $\mathbf{x}^\top A\mathbf{x}$ subject to $\|\mathbf{x}\|_2=1$. Give an example of $\mathbf{x}\in\mathbb{R}^3$ that attains the minimum.

4 (10) Let $A=A^\top\in M_{n\times n}(\mathbb{R})$ be a positive definite matrix and $\mathbf{x}\in\mathbb{R}^n$. Show that the matrix $\begin{bmatrix} A & \mathbf{x}\\ \mathbf{x}^\top & c\end{bmatrix}$ is positive definite if and only if $c>\mathbf{x}^\top A^{-1}\mathbf{x}$.

5 (24) Determine "true" or "false" for the following statements. Briefly sketch your proof when the answer is "true", or give a counterexample when the answer is "false".

a Suppose $A\in M_{3\times 4}(\mathbb{R})$ and $B\in M_{4\times 5}(\mathbb{R})$. If $AB=0$ then $\operatorname{rank}(A)+\operatorname{rank}(B)\le 4$.

b The inverse of a symmetric matrix is symmetric.

c If $A\in M_{3\times 3}(\mathbb{R})$ and $\operatorname{rank}(A)=1$ then A has a nonzero eigenvalue.

d Let $A,\ B\in M_{n\times n}(\mathbb{R})$. The determinant of $AB-BA$ is zero.