國立高雄大學九十七學年度研究所碩士班招生考試試題

科目：線性代數(甲)
考試時間：100 分鐘

系所：
應用數學系碩士班甲組
本科原始成績：100 分

是否使用計算機：是

Notations.

I_n : the identity matrix of size n.
$M_{n\times m}(\mathbb{R})$: set of $n \times m$ real matrices.
$A^\top$: the transpose of matrix A.
L_A: define $L_A : \mathbb{R}^m \to \mathbb{R}^n$ by $L_A(x) = Ax$, where $A \in M_{n\times m}(\mathbb{R})$.

1 a. (10) Let W_1 and W_2 be subspaces of a vector space V. Prove that the intersection $W_1 \cap W_2$ is a subspace of V.
b. (8) Let $W_1 = \text{span}\{(1,2,3),(2,1,1)\}$ and $W_2 = \text{span}\{(1,0,1),(3,0,-1)\}$. Find a basis for $W_1 \cap W_2$.

2 Let $A \in M_{3\times 3}(\mathbb{R})$ and $A^2 - 3A + 2I_2 = 0$ where 0 is a zero matrix.
a. (10) Show that A is diagonalizable.
b. (8) Describe all matrices A.

3 a. (5) Let $A \in M_{n\times n}(\mathbb{R})$ be an orthogonal matrix. Show that $\|Ax\|_2 = \|x\|_2$ for all $x \in \mathbb{R}^n$.
b. (8) Show that every 2×2 orthogonal matrix is one of two forms: either

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some angle θ.

4 Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{bmatrix}$ and $T = L_A$.
a. (10) Find the eigenvalues and the corresponding eigenvectors of A.
b. (8) Let $x \in \mathbb{R}^3$. Show that W=span$\{x, Ax, A^2x, ...\}$ is T-invariant subspace.
c. (5) Let $x = [15, -1, -13]^\top$. Find a basis for $W = \text{span}\{x, Ax, A^2x, ...\}$.
d. (8) Describe all T-invariant subspaces.

5 Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{(n+m)\times(n+m)}(\mathbb{R})$ where $A \in M_{n\times n}(\mathbb{R})$, $D \in M_{m\times m}(\mathbb{R})$ and $B, C^\top \in M_{n\times m}(\mathbb{R})$
a. (10) Show that, if A is invertible, then det(M)=det(A)det($D - CA^{-1}B$).
b. (10) Let $B \in M_{3\times 2}(\mathbb{R})$. Show that the matrix $\begin{bmatrix} I_3 & B \\ B^\top & -I_2 \end{bmatrix}$ is invertible.

國立高雄大學九十七學年度研究所碩士班招生考試試題

科目：高等微積分
考試時間：100 分鐘

系所：
應用數學系碩士班甲組
本科原始成績：100 分

是否使用計算機：是

1. (15%) (a) State the definition of compact set.
   (b) State the Heine-Borel Theorem.
   (c) Give an example of a bounded and closed set that is not compact (please, explain why).

2. (15%) If M is a metric space, prove that every compact subset of M is complete.

3. (15%) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$f(x, y, z) = (x + y + z, x - y - 2xz).$$

   Show that we can solve for $(x, y) = \varphi(z)$ near $z = 0$.

4. (15%) Prove that there is no continuous function taking $[0, 1]$ onto $(0, 1)$.

5. (15%)Let $f_n : [1, 2] \to \mathbb{R}$ be defined by $f_n(x) = x/(1+x)^n$.
   (a) Prove that $\sum_{n=1}^{\infty} f_n(x)$ is convergent for $x \in [1, 2]$;
   (b) Is it uniformly convergent?
   (c) Is $\int_1^2 \left(\sum_{n=1}^{\infty} f_n(x)\right) dx = \sum_{n=1}^{\infty} \int_1^2 f_n(x)\, dx$?

6. (15%) Evaluate the following double integral:

$$(a) \int_0^1 \int_y^1 e^{y/x} dxdy, \quad (b) \int_0^1 \int_0^{\sqrt{1-x^2}} \sin \sqrt{x^2 + y^2} dxdy.$$

7. (10%) Show that the cubic polynomials
   (a) $p(x) = x^3 + ax^2 + bx + c$ has no extreme values iff $a^2 \leq 3b$;
   (b) $q(x) = x^3 + ax + b$ has exactly one zero for $a > 0$.

國立高雄大學九十七學年度研究所碩士班招生考試試題

科目：線性代數(乙)　　系所：應用數學系碩士班乙組　　是否使用計算機：是
考試時間：100 分鐘　　本科原始成績：100 分

Notations.

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1 a. (10) Let W_1 and W_2 be subspaces of a vector space V. Prove that the intersection $W_1 \cap W_2$ is a subspace of V.
b. (8) Let $W_1 = \text{span}\{(1,2,3),(2,1,1)\}$ and $W_2 = \text{span}\{(1,0,1),(3,0,-1)\}$. Find a basis for $W_1 \cap W_2$.

2 Let $A \in M_{3\times 3}(\mathbb{R})$ and $A^2 - 3A + 2I_2 = 0$ where 0 is a zero matrix.
a. (10) Show that A is diagonalizable.
b. (8) Describe all matrices A.

3 a. (10) Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$. Find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.
b. (8) Let $A \in M_{n\times n}(\mathbb{R})$ and $A = QR$ where Q is orthogonal and R is upper triangular. Show that $B = RQ$ is similar to A.

4 Let $A = \begin{bmatrix} -1 & 3 & -3 \\ 0 & 5 & -6 \\ 0 & 3 & -4 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$.
a. (10) Find the eigenvalues and the corresponding eigenvectors of A.
b. (5) Find the general solution $\mathbf{x}_k$ of $\mathbf{x}_k = A\mathbf{x}_{k-1}$ starting from $\mathbf{x}_0$.
c. (8) Let $\mathbf{B} = \{\mathbf{x} | \|\mathbf{x}\|_2 = 1\}$ and let $F : \mathbf{B} \to \mathbf{B}$ be defined by $F(\mathbf{x}) = \frac{A\mathbf{x}}{\|A\mathbf{x}\|}$. Find $\lim_{n\to\infty} F^n(\mathbf{x}_0)$.

5 Let $\mathbf{x} \in \mathbb{R}^n$ and $A = I_n - \mathbf{x}\mathbf{x}^\top$.
a. (8) Find the eigenvalues and the corresponding eigenspaces of A.
b. (5) Show that $\det(A) = 1 - \mathbf{x}^\top\mathbf{x}$.
c. (10) Find A^{-1} if $\mathbf{x}^\top\mathbf{x} \neq 1$.

國立高雄大學九十七學年度研究所碩士班招生考試試題

科目：微分方程　　系所：應用數學系碩士班乙組　　是否使用計算機：是
考試時間：100 分鐘　　本科原始成績：100 分

Please work out all six problems

1. (32 points) Find the general solutions of the following differential equations:
   (a) $(4x+3y+1)dx+(x+y+1)dy=0$
   (b) $x\frac{dy}{dx}+xy=1-y,\quad y(1)=0$
   (c) $(x^2y^2-y)dx+(2x^3y+x)dy=0$
   (d) $4y''+y=2\sec(\frac{x}{2}),\quad -\pi<x<\pi.$

2. (a) (5 points) Find the inverse Laplace transform of $\frac{1}{\sqrt{s}(s-1)}$
   (b) (8 points) Find the solution of the equation

$$y'(t)+2y(t)+\int_0^t y(s)\,ds=H(t-1),\quad t>0,\quad y(0)=0,$$

   where $H(t)=1$, at $t\geq 0$, and $H(t)=0$, at $t<0$.

3. (a) (10 points) Find the Green's function for the boundary value problem

$$\begin{cases} y''+2y'+2y=0 \\ y(0)=y(\frac{\pi}{2})=0. \end{cases}$$

   (b) (5 points) Consider the boundary value problem $y''+\lambda y=0$, $0<x<1$, with $y(0)=0$ and $y(1)+2y'(1)=0$. Show that if y_m and y_n are eigenfunctions corresponding to the eigenvalues λ_n and λ_m, respectively, with $\lambda_n\neq\lambda_m$, then $\int_0^1 y_n(x)y_m(x)\,dx=0$.

4. (15 points) Find the general solution of $x^2y''-x(4-x)y'+(6-2x)y=0$ near the origin, $x>0$.

5. (10 points) Let $\mathbf{x_1},\ldots,\mathbf{x_n}$ be linearly independent solutions of the system $\mathbf{x}'(t)=\mathbf{A}(t)\mathbf{x}(t)$ for $t\in(a,b)$, where $\mathbf{A}(t)=\left[a_{i,j}(t)\right]_{n\times n}$. Let $W[\mathbf{x_1},\ldots,\mathbf{x_n}](t)$ be the determinant of the fundamental matrix $\left[\mathbf{x_1},\mathbf{x_2},\ldots,\mathbf{x_n}\right]$. Show that

$$\frac{dW[\mathbf{x_1},\ldots,\mathbf{x_n}](t)}{dt}=(a_{1,1}+\cdots+a_{n,n})W[\mathbf{x_1},\ldots,\mathbf{x_n}](t).$$

6. (15 points) Let $X(t)$ be the matrix function that satisfies

$$X'(t)=\begin{bmatrix}-3 & -1\\ 4 & 1\end{bmatrix}X(t),\quad X(0)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

   Find $X(t)$ and show that $X(t+s)=X(t)X(s)$, $t,s\geq 0$.