1. Given a sequence of continuous functions \( f_k : A \subset M \rightarrow N, M \text{ and } N \) being two metric spaces.

   (a) If \( f_k \rightarrow f \), is \( f \) continuous? Give a counter example if this is not true. \((6\%)\)

   (b) Prove that if \( f_k \rightarrow f \) uniformly on \( A \) then \( f \) is continuous on \( A \). \((13\%)\)

   (c) Find the limit function of the sequence \( f_k(x) = x^2 - x - 1 \) for \( 0 \leq x \leq 1 \) and determine the convergence of the sequence. \((5\%)\)

2. Given a function \( f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^m \) and \( x_0 \in A \).

   (a) Define the condition at which \( f \) is differentiable at \( x_0 \) and subsequently show that \( f(x, y) = (xy, y) \) is differentiable at \((x_0, y_0) \in A \subset \mathbb{R}^2\). \((10\%)\)

   (b) Alternatively, \( f \) is said to be differentiable at \( x_0 \) if given \( \epsilon > 0 \), \( \exists \delta > 0 \), such that for all \( ||x - x_0|| < \delta \) \( ||f(x) - f(x_0) - Df(x_0)(x - x_0)|| < \epsilon ||x - x_0|| \). Use this statement to show that if \( f \) is differentiable then \( x^2 f(x) \) is also differentiable at \( x_0 \) and the derivative is given by \( D(x^2 f(x_0)) = 2x_0 f(x_0) + x_0^2 Df(x_0) \). \((15\%)\)

3. Given \( f : A \rightarrow \mathbb{R}^n \). Let \( B \) be a rectangle containing \( A \), i.e. \( A \subset B \) and \( \mathcal{P}(B) \) is the set of all possible partitions of \( B \). Let \( L(f, P) \) and \( U(f, P) \) denote the upper and lower sum of \( f \) and define \( s = \sup \{ L(f, P) | P \in \mathcal{P}(B) \} \) and \( S = \inf \{ U(f, P) | P \in \mathcal{P}(B) \} \).

   (a) Define the condition at which \( f \) is Riemann integrable and subsequently use the definition to evaluate the integral \( \int_0^3 (x + 5) \, dx \). \((10\%)\)

   (b) Alternatively, \( f \) is said to be integrable i.e. for any \( \epsilon > 0 \) there exists a partition \( P_\epsilon \) of \( B \) such that \( 0 \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon \). Use this statement to show that if \( f \) is continuous and \( A = [a, b] \) then \( f \) is integrable. \((15\%)\)

4. Let \( f : A \subset \mathbb{R}^3 \rightarrow \mathbb{R} \) be differentiable on \( A \) and \( f(x, y, z) = x^2 + y^2 + z^2 + xy + yz + zx \).

   (a) Find the critical point of \( f \) and determine the nature of it. \((10\%)\)

   (b) Find the critical point of \( f \) subject to the constraint \( g(x, y, z) = x + y + z \) with \( \{ (x, y, z) \in \mathbb{R}^3 | g(x, y, z) = 1 \} \). \((5\%)\)

   (c) Let \( S \) be the surface \( f(x, y, z) = 2 \). Find the unit normal to the surface \( S \) at \( c = (1, -1, 1) \) and subsequently determine the tangent plane to the surface at \( c \). \((5\%)\)

   (d) What is the directional derivative of \( f \) at \( c \) in the direction that the rate of change is the greatest? \((5\%)\)
Notations.

- $I_n$: the identity matrix of size $n$.
- $[T]_\beta^\gamma$: matrix representation of $T$ relative to ordered bases $\beta$, $\gamma$.
- $M_{n \times m}(\mathbb{R})$: set of $n \times m$ real matrices.
- $A^\top$: the transpose of matrix $A$.

1. Let $H = \text{span}\{(1, -1, 0, 2)^\top, (2, 1, -2, 0)^\top, (0, -3, 2, 4)^\top, (3, 3, -4, -2)^\top\}$.
   a. (6) Find a basis for $H$.
   b. (10) Which vector in $H$ is closed to $(7, 5, 0, 3)^\top$?

2. (10) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear and let $\beta = \{(1, 1), (0, 1)\}$
   $\gamma = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ be ordered bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.
   If
   
   \[
   [T]_\beta^\gamma = \begin{bmatrix}
   1 & 2 \\
   1 & 3 \\
   0 & 1
   \end{bmatrix}
   \]
   
   then find a matrix $A$ such that $T(x) = Ax$.

3. Let $v$ be a unit vector in $\mathbb{R}^n$ and $A = I_n - 2vv^\top$
   a. (6) Show that $\|Ax\|_2 = \|x\|_2$ for all $x \in \mathbb{R}^n$.
   b. (6) Find the eigenvalues and corresponding to eigenspaces of $A$.
   c. (10) Let $x = (1, 1, 0)^\top$. Find a vector $v \in \mathbb{R}^3$ such that $Ax = (\sqrt{2}, 0, 0)^\top$.

4. Given

   \[
   A = \begin{bmatrix}
   3 & 0 & 1 \\
   0 & 3 & 1 \\
   0 & -1 & 1
   \end{bmatrix}
   \]

   a. (10) Find the Jordan canonical form $J$ of $A$ and a matrix $Q$ such that $Q^{-1}AQ = J$.
   b. (3) Find the minimal polynomial of $A$.
   c. (5) Find scalars $a, b, c \in \mathbb{R}$ such that $A^{-1} = aI_3 + bA + cA^2$.

5. (10) Suppose $A = A^\top \in M_{n \times n}(\mathbb{R})$. If $\lambda_1$ and $\lambda_0$ are distinct eigenvalues of $A$ with corresponding eigenvectors $x_1$ and $x_2$ then show that $x_1$ and $x_2$ are orthogonal.

6. (24) Determine "true" or "false" for the following statements. Briefly sketch your proof when the answer is "true", or give a counterexample when the answer is "false".

   a. The intersection of two subspaces of a vector space is a subspace.
   b. If $A \in M_{4 \times 4}(\mathbb{R})$ and $A^2 + 3A + 2I = 0$ where $0$ is a zero matrix, then $A$ is invertible.
   c. If $A \in M_{3 \times 3}(\mathbb{R})$ and $\text{rank}(A) = 1$ then $A$ has a nonzero eigenvalue.
   d. Let $A, B \in M_{n \times n}(\mathbb{R})$. The determinant of $AB - BA$ is zero.
1. (30%) Find the general solution of the following differential equations:
   (a) $e^y \, dx + (xe^y + 2y) \, dy = 0$.
   (b) $y \, dx + (x^2y - x) \, dy = 0$.
   (c) $y'' + y = \sin x$.

2. (15%) Verify the $y_1(x) = x$ and $y_2(x) = 1/x$ are solutions of
   
   $x^2y'' + xy' - y = 0$,
   
   and then find the general solution of
   
   $x^2y'' + xy' - y = x \ln x$.

3. (20%) Consider the homogeneous system of linear differential equations
   
   $X'(t) = AX(t)$, where
   
   $X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$.

   (a) Find the eigenvalues and eigenvectors of $A$.
   (b) Find the general solution of the system.
   (c) Classify the critical point $(0, 0)$ as to type and determine whether it is stable, asymptotically stable, or unstable.

4. (20%) Consider the Sturm–Liouville problem
   
   $y'' + \lambda r(x) y = 0$, $x \in (0, 1)$

   with the boundary condition
   
   $a_1 y(0) + a_2 y'(0) = 0$ and $b_1 y(1) + b_2 y'(1) = 0$,

   where $r(x)$ is a positive continuous function on the interval $[0, 1]$ and $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

   (a) Show that all eigenvalues $\lambda$ are real.
   (b) Suppose that $\phi_1$ and $\phi_2$ are two eigenfunctions of problem corresponding to eigenvalues $\lambda_1$ and $\lambda_2$, respectively. Show that if $\lambda_1 \neq \lambda_2$, then
   
   $\int_0^1 r(x) \phi_1(x) \phi_2(x) \, dx = 0$.

5. (15%) Use the Laplace transform to solve the differential equation
   
   $y'' + 4y = 4x$

   that satisfies the initial conditions $y(0) = 1$ and $y'(0) = 5$. 
Notations.

$I_n$: the identity matrix of size $n$.
$M_{n\times m}(\mathbb{R})$: set of $n \times m$ real matrices.
$A^\top$: the transpose of matrix $A$.

1. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \\ -2 & 0 & 6 \\ -1 & 4 & 5 \end{bmatrix}.$$  

a. (15) Find a basis for the row space, a basis for the column space and a basis for the null space of the matrix.
b. (8) Which vector in column space of $A$ is closed to $(4, 2, 6, 6)^\top$.

2. Let $v$ be a unit vector in $\mathbb{R}^n$ and $A = I_n - 2vv^\top$.
a. (6) Show that $\|Ax\|_2 = \|x\|_2$ for all $x \in \mathbb{R}^n$.
b. (6) Find the eigenvalues and corresponding eigenspaces of $A$.
c. (10) Let $x = (1, 1, 0)^\top$. Find a vector $v \in \mathbb{R}^3$ such that $Av = (\sqrt{2}, 0, 0)^\top$.

3. Given

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$  

a. (8) Find the eigenvalues and corresponding eigenvectors of $A$.
b. (3) Find the minimal polynomial of $A$.
c. (5) Find scalars $a, b \in \mathbb{R}$ such that $A^{-1} = aI + bA$.
d. (5) Find the minimum of $x^\top Ax$ subject to $\|x\|_2 = 1$. Give an example of $x \in \mathbb{R}^3$ that attains the minimum.

4. (10) Let $A = A^\top \in M_{n\times n}(\mathbb{R})$ be a positive definite matrix and $x \in \mathbb{R}^n$. Show that the matrix $\begin{bmatrix} A & x \\ x^\top & c \end{bmatrix}$ is positive definite if and only if $c > x^\top A^{-1}x$.

5. (24) Determine “true” or “false” for the following statements. Briefly sketch your proof when the answer is “true”, or give a counterexample when the answer is “false”.

a. Suppose $A \in M_{3\times 4}(\mathbb{R})$ and $D \in M_{4\times 5}(\mathbb{R})$. If $AD = 0$ then $\text{rank}(A) + \text{rank}(D) \leq 4$.
b. The inverse of a symmetric matrix is symmetric.
c. If $A \in M_{3\times 3}(\mathbb{R})$ and $\text{rank}(A) = 1$ then $A$ has a nonzero eigenvalue.
d. Let $A, B \in M_{n\times n}(\mathbb{R})$. The determinant of $AB - BA$ is zero.