1. (a) Find the eigenvalues and eigenfunctions of problem:
\[ y'' + \lambda y = 0, \quad y(-\pi/2) = 0, \quad y(\pi/2) = 0. \] (8%)

(b) The differential equation \( (1 - x^2)y'' - 2xy' = 0 \) has one basis solution \( y_1 = 1 \).
Show that the other basis solution on the interval \(-1 < x < 1\) is \( y_2 = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) \). (10%)

2. A forced oscillator is governed by
\[ y'' + \omega^2 y = r(t), \quad y(0) = A, \quad y'(0) = B, \]
where \( r(t) \) is the unspecified driving force. Show that the solution can be expressed as in
convolution form \( y(t) = A \cos(\omega t) + \frac{B}{\omega} \sin(\omega t) + \frac{1}{\omega} \sin(\omega t) * r(t) \). (10%)

3. Find a basis of eigenvectors and then diagonalize the following matrix (10%)
\[
\begin{bmatrix}
-8 & 11 & 3 \\
4 & -1 & 3 \\
-4 & 10 & 6
\end{bmatrix}
\]

4. Write down the (a) Divergence Theorem, (b) Stokes's Theorem, and (c) Green's Theorem in the Plane. (d) Laplace's Equation, \( \nabla^2 \phi = 0 \), in Cylindrical Coordinate. (12%)

5. Find the Fourier transform of the function \( e^{-ax^2}, \) where \( a > 0 \). (10%)

6. Consider semi-infinite string that satisfies the wave equation \( \frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \), \( c \) is a constant.
This elastic string also satisfies the following conditions: (i) It is initially at rest on the x-axis from \( x = 0 \) to \( \infty \). (ii) For time \( t > 0 \) the left end of the string is moved in a given fashion, namely, \( w(0,t) = f(t) = \begin{cases} \sin t & 0 \leq t \leq 2\pi, \\ 0 & \text{otherwise}. \end{cases} \) (iii) Furthermore \( \lim_{x \to \infty} w(x,t) = 0 \) for \( t \geq 0 \).
Please use the Laplace transform to solve the wave equation. (10%)

7. Find the integral (a) \( \int_{-\infty}^{\infty} \frac{1}{1+4x^2} \, dx \), (b) \( \int_{0}^{\infty} \frac{\sin x}{x} \, dx \). (20%)

8. (i) Show that \( z = \pm i \) are the branch points of \( f(z) = \sqrt{1+z^2} \). (ii) Plot and discuss the possible branch cut(s) of this complex function. (10%)
1. The Earth has moved around the Sun $5 \times 10^{10}$ years with velocity $v = 3 \times 10^4 \text{ m/s}$, what is the time difference between two clocks in the Earth and the Sun. (10%)

2. Two spin one half particles ($S_1$ and $S_2$) interact to each other, the Hamiltonian is $H = S_1 \cdot S_2$, what is the ground state and ground state energy for $J > 0$ and $J < 0$, respectively. (10%)

3. $H = H_0 + H_1$, $H_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and perturbation interaction $H_1 = \begin{pmatrix} \varepsilon & \varepsilon & 0 \\ \varepsilon & 0 & \varepsilon \\ 0 & \varepsilon & 0 \end{pmatrix}$ with $\varepsilon \ll 1$. Please calculate the Eigen value to second order and Eigen function to first order. (15%)

4. A three dimensional simple harmonic oscillator has the Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}k(x^2 + y^2 + z^2)$, please write down the degeneracy number of the states from the first to the forth excitations. (10%)

5. The Hamiltonian for an axially symmetric rotator with angular momentum $L = 1$ is $H = \frac{L_x^2}{2l_1} + \frac{L_y^2}{2l_2} + \frac{L_z^2}{2l_3}$, what are the Eigen values of $H$. (10%)

6. An electron is confined to a one-dimensional infinite potential well of width $a$. Calculate the expectation values of $x$ and $x^2$ for the ground state. (15%)

7. The one dimensional harmonic oscillator, $H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2}kx^2$, its ground state trial wave function is $\varphi(x) = A \exp(-Bx^2)$. What is the value of $B$? (10%)

8. The one dimensional quantum system has a delta potential well, $V(x) = -aV_0 \delta(x)$, what is the bond state energy? (10%)

9. Radiation of wavelength $\lambda = 290 \text{ nm}$ falls on a metal surface for which the work function is $W = 4.05 \text{ eV}$. What potential is needed to stop the most energetic photoelectrons? ($h = 6.63 \times 10^{-34} \text{ J s}$) (10%)
1. (25%) Find the interior and exterior magnetic fields of an ideal solenoid with radius \( R \), length \( L \) and \( N \) turns while a current \( i \) is applied.

2. (30%) Describe Hall effect. What kind of physical quantities can be obtained from Hall effect? Describe their significance.

3. (25%) Describe the Young's interference experiment and interpret its physical meaning/significance.

4. (20%) Give physical definitions to conductor, insulator, semiconductor, and superconductor.
1. Briefly give the definition of following terms, please deliver relation formula as if need:
   (a). Channel length modulation effect (3%)
   (b). Show four amplifier types, including circuit model, gain, ideal characterization of input/output resistance (16%)
   (c). Transmission frequency. (3%)
   (d). Concentration profile of the minority carriers of pnp transistor operating in active mode. (5%)

2. Fig.1 shows a discrete-circuit CS amplifier. Answer following questions. (In figure, capacitors are very large, shown as infinite)
   (a). If the transistor has $V_t=1V$, and $k_n W/L=2mA/V^2$, verify that the bias circuit establishes $V_{gs}=2V$, $I_D=1mA$, and $V_D=+7.5V$. That is, assume these values, and verify that they are consistent with the values of the circuit components and the device parameters. (10%)
   (b). Find $g_m$ and $r_0$ if $V_A=100V$. (6%)
   (c). Draw a complete small-signal equivalent circuit for the amplifier assuming all capacitors behave as short circuit at signal frequencies. (6%)
   (d). Find $R_{in}$, $v_{gs}/v_{sig}$, $v_{o}/v_{gs}$, and $v_{o}/v_{sig}$. (16%)

3. Fig.2 shows a transistor amplifier, please determine its voltage gain. Assume $\beta = 100$. (20%)

4. Comparison of the MOSFET and BJT, briefly answer following questions.
   (a). Circuit Symbol. (2%)
   (b). DC bias for active mode. (2%)
   (c). current-voltage characterization formula in the active region. (3%)
   (d). Low-frequency hybrid-$\pi$ model. (4%)
   (e). High-frequency Model. (4%)

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![Fig. 1](image1)

![Fig. 2](image2)
The electromagnetic wave travels from medium 1 (permeability \( \mu_1 \), refraction index \( n_1 \)) to medium 2 (permeability \( \mu_2 \), refraction index \( n_2 \)) with oblique incidence. The xy plane forms the boundary between the two linear non-conducting media. The Fresnel’s equations for the case of polarization in the plane of incidence are

\[
\tilde{E}_{\alpha} = (a - b) \tilde{E}_{\alpha}^0 - \frac{2}{a + b} \tilde{E}_{\alpha}^0 \\
\tilde{E}_{\beta} = (a + b) \tilde{E}_{\beta}^0 \\
\tilde{E}_{\gamma} = (a + b) \tilde{E}_{\gamma}^0 \\
\tilde{E}_{\delta} = (a - b) \tilde{E}_{\delta}^0
\]

where \( \tilde{E}_{\alpha}^0 \), \( \tilde{E}_{\beta}^0 \), and \( \tilde{E}_{\gamma}^0 \) are the complex amplitudes of reflected, transmitted, and incident waves, respectively, and

\[
a = \frac{\cos q_1}{\cos q_2}, \quad b = \frac{m_2}{m_1}.
\]

(a). Calculate the Brewster’s angle \( \theta_B \) if \( \mu_1 \sim \mu_2 \).

(b). Calculate the reflection and transmission coefficients, and then prove the energy conservation.

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A uniformly magnetized sphere of radius \( R \) with magnetization \( \hat{M} = M \hat{z} \) is situated at the origin. Answer the following questions:

(a). Calculate the magnetic induction \( \hat{B} \) inside the sphere. (Hint: The magnetic induction \( \hat{B} \) inside a spherical shell, of radius \( R \), carrying a uniform surface charge \( \sigma \) and spinning at angular velocity \( \omega \), is \( \hat{B} = \frac{2}{3} m \sigma s R \omega \)).

(b). Calculate the magnetic induction \( \hat{B} \) at point \((0,0,2R)\) in terms of \( \hat{M} \). (Hint: The magnetic induction \( \hat{B} \) caused by a magnetic dipole \( \hat{r} \) is \( \hat{B} = \frac{m}{4\pi} \frac{1}{r^3} [3(\hat{r} \cdot \hat{m}) \hat{r} - \hat{m}] \)).
(c). If now the measured magnetic induction and auxiliary field inside the sphere are $\vec{B}_0$ and $\vec{H}_0$, respectively, and then a spherical cavity is hollowed out of that sphere. Find the auxiliary field $\vec{H}$ at the center of the cavity in terms of $\vec{H}_0$ and $\vec{M}$.

三. (24 Points) If we consider the monochromatic electromagnetic wave confined to the interior of a rectangular wave guide (assumed to be a perfect conductor and $a > b$) and propagated along the z axis as shown in the Fig. 1, the fields can be expressed as

$$\vec{E}(x, y, z, t) = \vec{E}_0(x, y)e^{i(kz - \omega t)}, \quad \vec{B}(x, y, z, t) = \vec{B}_0(x, y)e^{i(kz - \omega t)}$$

where $\vec{E}_0 = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$ and $\vec{B}_0 = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$, and also

$$E_x = \frac{i}{(w/c)^2 - k^2} \left( k \frac{\partial E_z}{\partial x} + w \frac{\partial B_z}{\partial y} \right) \quad E_y = \frac{i}{(w/c)^2 - k^2} \left( k \frac{\partial E_z}{\partial y} - w \frac{\partial B_z}{\partial x} \right)$$

$$B_x = \frac{i}{(w/c)^2 - k^2} \left( k \frac{\partial B_z}{\partial x} - \frac{w}{c} \frac{\partial E_z}{\partial y} \right) \quad B_y = \frac{i}{(w/c)^2 - k^2} \left( k \frac{\partial B_z}{\partial y} + \frac{w}{c} \frac{\partial E_z}{\partial x} \right)$$

The uncoupled equations for $E_z$ and $B_z$ are:

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (w/c)^2 - k^2 \right] E_z = 0 \quad \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (w/c)^2 - k^2 \right] B_z = 0$$

(a). Calculate the $E_z(x, y)$ solution of $TM_{mn}$ mode.

(b). Find the lowest $TM$ cutoff frequency for this wave guide.

![Fig. 1](image)
四. (20 Points) A sphere of homogeneous linear dielectric material with the electric susceptibility $\varepsilon_e$ is placed in an uniformly external electric field $E_0$.

(a). Find the electric potential and electric field inside the sphere by solving Laplace's equation.

(b). Find the electric field produced solely by the polarization $\mathbf{P}$ inside the sphere.

五. (16 Points) The magnetic field at a distant $s$ from a straight segment of wire carrying a steady current $I$ is

$$B = \frac{\mu_0 I}{4\pi s} \left( \sin \theta_2 - \sin \theta_1 \right),$$

where $\theta_1$ and $\theta_2$ are the initial and final angles, respectively.

(a). Find the magnetic field at the center of a square loop, which carries a steady current $I$. Let $R$ be the distance from center to side, as shown in Fig. 2(a).

(b). If an infinite straight wire carrying the same steady current $I$ is placed near that square loop, as shown in Fig. 2(b). Find the force on that square loop.

![Fig. 2](image-url)

(a) (b)