1. (8%) Find the limit (if it exists)
\[
\lim_{n \to \infty} (1 + a^2)(1 + a^4) \cdots (1 + a^{2^n})
\]

2. (12%) Find the limit
\[
\lim_{n \to \infty} \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n(n + 1)(n + 2)}
\]

3. (10%) Show that the equation \(5ax^4 + 2bx + c = 0\) has at least one zero in \((0, 1)\).

4. Evaluate the following integrals
   
   (a) (8%) \[ \int_{0}^{\pi/2} \frac{\tan^3 x}{\tan^3 x + \cot^3 x} \, dx \]
   
   (b) (12%) \[ \int_{0}^{3} \int_{3y}^{2y} \frac{y}{\sqrt{16 + x^2}} \, dx \, dy \]

5. Let \(C^\infty(\mathbb{R})\) be the space of infinitely differentiable functions on \(\mathbb{R}\) and \(T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})\) be defined by \(T(f) = f + f'\). Denote \(P_2(\mathbb{R}) = \{a + bx + cx^2 : a, b, c \in \mathbb{R}\}\).
   
   (a) (5%) Find the null space \(N(T)\) of \(T\). Is \(T\) invertible?
   
   (b) (5%) Show that \(T(P_2(\mathbb{R})) \subseteq P_2(\mathbb{R})\), i.e., \(P_2(\mathbb{R})\) is \(T\)-invariant.
   
   (c) (5%) Let \(S\) be the restriction of \(T\) on \(P_2(\mathbb{R})\). Find \([S]_\beta\), the matrix representation of \(S\) with respect to the ordered basis \(\beta = \{1, x, x^2\}\).
   
   (d) (10%) Is \(S\) invertible? If your answer is "yes", find \([S]_\beta^{-1}\).

6. For each of the following matrices \(A \in M_{n \times n}\), test \(A\) for diagonalizability and then find \(e^A\)

   (a) (10%) \[ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \]
   
   (b) (15%) \[ A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix} \]
(1) Suppose \( \left( X_1, \ldots, X_n \right) \) be a random sample with p.d.f \( f(x) = \lambda \exp\{-\lambda x\} \) where \( x \geq 0, \lambda > 0 \).

(a) Show that \( T(X) = \sum_{i=1}^{n} X_i \) is a complete sufficient statistic of \( \lambda \); (5%)

(b) Find an UMVUE of \( \theta = \exp\{-\lambda\} \); (10%)

(c) Use the Delta Method to derive a 95% confidence interval of \( \theta \). (10%)

(2) Let \( X_1, \ldots, X_n \) be i.i.d Poisson(\( \lambda \)).

(a) Find the UMP, size \( \alpha \), test of \( H_0 : \lambda = \lambda_0 \) vs \( H_a : \lambda > \lambda_0 \); (10%)

(b) Show that there does not exist a UMP, size \( \alpha \), test of \( H_0 : \lambda = \lambda_0 \) vs \( H_a : \lambda \neq \lambda_0 \); (10%)

(c) Consider the specific case \( H_0 : \lambda \leq 1 \) vs \( H_a : \lambda > 1 \), use the Central Limit Theorem to determine the sample size \( n \) so that the UMP test satisfies \( P(\text{reject } H_0 | \lambda = 1) = 0.05 \) and \( P(\text{reject } H_0 | \lambda = 2) = 0.95 \). (Recall: \( \varphi(1.64) = 0.95 \) ) (10%)

(3) Let \( X_1, \ldots, X_n \) be a random sample from the uniform distribution on the interval \([-1, 0]\).

(a) Find a minimal sufficient statistic of \( \theta \); (5%)

(b) Find an UMVUE of \( \theta \); (5%)

(c) Compare the variance of the UMVUE with the Cramer-Rao lower bound for the variance of an unbiased estimator and explain why this lower bound is not applicable in this instance. (5%)

(4) Let \( X \) and \( Y \) be i.i.d N(0,1) random variables, and define \( Z = \min(X, Y) \). Prove that \( Z^2 \) has chi-square distribution with degree of freedom 1. (15%)

(5) Suppose that one has \( n \) random pairs of measurements \( \left( X_i, Y_i \right) \) \( (i = 1, \ldots, n) \) with joint p.d.f.

\[
f_{X,Y}(x,y) = \lambda_i \tau_i \exp\{-\lambda_i x - \tau_i y\} \quad \text{where} \quad \lambda_i, \tau_i > 0 \quad (i = 1, \ldots, n) \quad \text{are 2n unknown positive constants.}
\]

Assume that the \( n \) pair unknown constants \( \left( \lambda_i, \tau_i \right) \) \( (i = 1, \ldots, n) \) lie on a straight line which pass through the origin. It is required to estimate the slope of that straight line. Obtain a maximum likelihood solution of this problem, and elaborate the computational details. (15%)
1. Prove each of the following statements. (Assume that any conditioning event has positive probability)

(1) If \( P(B) = 1 \), then \( P(A|B) = P(A) \) for any \( A \). (5%)

(2) Assume that \( P(A) > 0 \) and \( P(B) > 0 \). If \( A \) and \( B \) are independent, then they cannot be mutually exclusive, and if \( A \) and \( B \) are mutually exclusive, then they cannot be independent. (5%)

2. Prove that the following functions are cdfs.

(1) \( 1 - \exp(-x), x \in (0, \infty) \),
(2) \( \exp(-e^{-x}), x \in (-\infty, \infty) \). (10%)

3. Consider the following two pdfs,

\[
    f_1(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \log x \right), x \geq 0, \text{ and } f_2(x) = f_1(x)\left[1 + \sin(2\pi \log x)\right], x \geq 0.
\]

Show that

(1) If \( X_1 \sim f_1(x) \), then \( E(X_1^r) = \exp(r^2/2), r = 0, 1, 2, \ldots \) (10%)

(2) Suppose \( X_2 \sim f_2(x) \). Then \( E(X_1^r) = E(X_2^r) \) for \( r = 0, 1, 2, \ldots \). (10%)

4. Prove the following inequalities.

(1) Let \( X \) be a random variable with moment-generating function, \( M_X(t) \), \(-h < t < h\). Show that

\[
    P(X \geq a) \leq e^{-at}M_X(t), 0 < t < h, \text{ and } P(X \leq a) \leq e^{at}M_X(t), -h < t < 0.\]

(10%)

(2) Let \( X_1, \ldots, X_n \) be iid with mgf \( M_X(t) \), \(-h < t < h\), and let \( S_n = \sum_{i=1}^{n} X_i \). Show that

\[
    P(S_n > a) \leq e^{-at}[M_X(t)]^n, 0 < t < h, \text{ and } P(S_n \leq a) \leq e^{-at}[M_X(t)]^n, -h < t < 0.\]

(10%)
5. Suppose $U_1$ and $U_2$ are iid Uniform(0,1). Let

$$X_1 = \cos(2\pi U_1) \sqrt{-2 \log U_2} \text{ and } X_2 = \sin(2\pi U_1) \sqrt{-2 \log U_2}.$$ 

Prove that $X_1$ and $X_2$ are independent $N(0,1)$ random variables. (10%)

6. Let $X$ and $Y$ be independent $N(0,1)$ random variables. Define a new random variable, $Z$, by

$$Z = \begin{cases} X, & \text{if } XY > 0 \\ -X, & \text{if } XY < 0 \end{cases}.$$

Show that $Z$ has a normal distribution. (10%)

7. Find the pdf of $\prod_{i=1}^n X_i$, where $X_i$s are independent Uniform(0,1) random variables. (10%)

8. Let $X \sim \text{Poisson}(\theta)$ and $Y \sim \text{Poisson}(\lambda)$, and $X$ and $Y$ are independent. Find the distribution of $X | X + Y$. (10%)