1. A *lower triangular matrix* $A$ is an $n$-by-$n$ array in which $a_{ij} = 0$, if $i < j$.
   (1) What is the maximum number of nonzero elements in such an array? (5%)
   (2) Assume the nonzero elements of the matrix $A$ is represented sequentially in an
       one-dimensional array $b$ by row major, e.g.,

       \[
       \begin{pmatrix}
       a_{11} & 0 & 0 \\
       a_{21} & a_{22} & 0 \\
       a_{31} & a_{32} & a_{33} \\
       a_{41} & a_{42} & a_{43} & a_{44}
       \end{pmatrix}
       \]

       matrix $A$ | $a_{11}$ $a_{21}$ $a_{22}$ $a_{31}$ $a_{32}$ $a_{33}$ ...

       What is the position in array $b$ that stores element $a_{ij}$? (10%)

2. Consider the following recursive function.

   $A(m, n) = \begin{cases} 
   m + 1, & \text{if } n = 0 \\
   A(m, n - 1), & \text{otherwise}
   \end{cases}$

   (1) What is the result of $A(3, 2)$? (5%)
   (2) Write a recursive algorithm to complete this function. (10%)

3. Consider an array $NODE$ that consists of the linked list pointed to by $List$ and another available
   list of unused nodes pointed to by $Avail$.

   \[
   \begin{array}{c|c|c}
   0 & BOB & 7 \\
   1 & 2 \\
   2 & 6 \\
   3 & LIN & 5 \\
   4 & 1 \\
   5 & ROSE & -1 \\
   6 & 8 \\
   7 & JOE & 3 \\
   8 & -1 \\
   \end{array}
   \]

   (1) List the elements of the linked list, starting from $List$. (6%)
   (2) Show the resulting array $NODE$ after inserting MARY into the linked list starting from $List$
       so that the resulting list is still in alphabetical order? (7%)
   (3) After inserting MARY, if LIN is removed, what is the resulting array $NODE$? (7%)
4. Consider the following binary search tree.

(1) What are the resulting sequences obtained by traversing the tree in inorder and postorder? (10%)
(2) Please show how to represent the binary tree with a sequential array. (10%)
(3) Devise an algorithm that can compute the number of leaf nodes for a given binary tree. (10%)

```
A
 /   \
B     F
 /     /
C   G   H
   /
  D
```

5. A “ternary” search algorithm is a divide-and-conquer searching approach which first tests the element at position \( \frac{n}{3} \) for equality with some value \( x \) and then possibly checks the element at \( 2 \cdot \frac{n}{3} \) either discovering \( x \) or reducing the set size to one third of the original.

(1) Show the searching steps for searching \( x = 21 \) over the following sorted sequences. (10%)

```
1  3  5  7  9  11  13  15  17  19  21  23  25  27  29
```

(2) What is the time complexity of this algorithm? You have to show the derivation. (10%)
1. Consider the three persons, John, George, and Peter, and 8 different balls, \( b_1, b_2, b_3, b_4, b_5, \\
\quad b_6, b_7, \) and \( b_8 \).

(a) (4%) In how many ways can these 8 balls be arranged in a line so that \( b_1 \) is not at position 1
and \( b_2 \) is not at position 2?

(b) (4%) In how many ways can these 8 balls be arranged in a line so that \( b_i \) is not at position \( i \),
\( i = 1, 2, 3, \ldots, 8 \)?

(c) (4%) In how many ways can these 8 balls be distributed to the three persons?

(d) (4%) In how many ways can these 8 balls be distributed to the three persons such that each
person gets at least one ball?

(e) (4%) In how many ways can these 8 balls be distributed to the three persons such that John
does not get \( b_1 \), George does not get \( b_2 \), and Peter does not get \( b_3 \)?

(f) (4%) In how many ways can these 8 balls be packed into 3 same boxes so that each box
contains at least one ball?

(g) (4%) In how many ways can these 8 balls be packed into 3 same boxes (boxes can be empty)?

2. In a shop, there are four kinds of flags, red, green, blue, and yellow.

(a) (4%) In how many ways can John pick 8 flags?

(b) (4%) In how many ways can John pick 8 flags so that each kind of flag is selected at least
one?

(c) (4%) In how many ways can John pick 8 flags so that each kind of flag is selected at most
three?

(d) (4%) In how many ways can John pick 8 flags so that he selects an even number of red flags,
an odd number of green flags, and any number of blue and yellow flags?

(e) (4%) In how many ways can John pick 8 flags so that the number of red flags is less than or
equal to the number of green flags, and any number of blue and yellow flags?

3. Let \( A = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \).

(a) (5%) Compute \( A^2, A^3, A^4, \) and \( A^5 \).

(b) (5%) Conjecture a general formula for \( A^n, \ n \in \mathbb{Z}^+ \), and establish your conjecture by the
Principle of Mathematical Induction.

4. (8%) \( 42x + 90y = c, \ x, \ y \in \mathbb{Z}, \ 0 < c < 10, \ c \in \mathbb{Z}^+. \) Find all solutions of \( x, \ y, \) and \( c. \)
5. What is the number of ways to properly color the vertices of the following figures using 6 colors so that if \( \{a, b\} \) is an edge, then \( a \) and \( b \) are colored with different colors?

(a) (5%)  
(b) (5%)

\[
\begin{array}{c}
\text{a} \\
\text{c} \\
\text{b} \\
\text{c}
\end{array}
\]

6. (8%) Prove that for all integers \( x, y, \) and \( z \), if \( x + y + z \geq 0 \), then \( x \geq 0 \) or \( y \geq 0 \) or \( z \geq 0 \).

7. (8%) For \( n \in \mathbb{Z}^+ \), prove that \( 3 \mid (n^3 + 2n) \).

8. (8%) Let \( \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \subseteq \mathbb{Z}^+ \). Show that for some \( i \neq j \), \( x_i - x_j \) is divisible by 7.