Chapter I Combinatorial Analysis

In many experiments with finite possible results, such as tossing one die, it may be reasonable to assume that all the possible results are equally likely. In that case, a realistic probability model should be solved by simply counting the number of different ways that a certain event can occur. The mathematical theory of counting is formally known as ***combinatorial analysis***.

Principle of Counting: If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes, and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment, and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of third experiment, and if,…, then there are a total of $n_1 \times n_2 \times \cdots \times n_r$ possible outcomes of the r experiments.

Example 1-1: How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers? How many license plates would be possible if repetition among letters or numbers are prohibited?

Solution: (a) 26*26*26*10*10*10*10 = 175,760,000.

(b) 26*25*24*10*9*8*7 = 78,624,000. ◘

Example 1-2: How many functions defined on n points are possible if each functional value is either 0 or 1? **Solution:** 2^n. ◘

Permutation is the ***ordered*** arrangements of a set of n objects. How many different ordered arrangements of a set of n objects? There are $n(n-1)(n-2)\cdots(3)(2)(1) = n!$ possible orders. We might want to determine the number of permutations of n objects, of which n_1 are alike, n_2 are alike,…, n_r are alike, given that $n_1 + n_2 + \cdots + n_r = n$. There are $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$ different permutations.

We are often interested in determining the number of different ***groups*** of r objects that could be formed from a total of n objects (**combinations**). If the order is relevant, say permutations, there are $n(n-1)(n-2)\cdots(n-r+1)$ possible permutations. Since every group consists of the same items will be counted $r(r-1)(r-2)\cdots(2)(1) = r!$ times, it follows that the total number of groups that can be formed is $\dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \dfrac{n!}{(n-r)!\,r!}$, denoted by $\dbinom{n}{r}$.

A useful combinatorial identity is $\dbinom{n}{r} = \dbinom{n-1}{r-1} + \dbinom{n-1}{r}$ $\quad 1 \le r \le n$. It can be proved by combinatorial argument. Consider a group of n objects and fix attention on some particular one of these objects—call it object 1. Now, there are $\binom{n-1}{r-1}$ combinations of size r that contain object 1. Also, there are $\binom{n-1}{r}$ combinations of size r that do not contain object 1. As there are a total of $\binom{n}{r}$ combinations of size r. The value $\binom{n}{r}$ are often referred to as binomial coefficients.

***<u>Binomial theorem</u>*:** $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. (1.1)

Proof by Induction: When $n = 1$, Equation (1.1) reduces to $x + y = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0$.

Assume Equation (1.1) for $n-1$. Now

$$\begin{aligned}
(x+y)^n &= (x+y)(x+y)^{n-1} \\
&= (x+y)\sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k} .
\end{aligned}$$

Letting $i = k+1$ in the first sum and $i = k$ in the second sum, we find that

$$\begin{aligned}
(x+y)^n &= \sum_{i=1}^{n} \binom{n-1}{i-1} x^i y^{n-i} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\
&= x^n + \sum_{i=1}^{n-1} \left[\binom{n-1}{i-1} + \binom{n-1}{i} \right] x^i y^{n-i} + y^n \\
&= x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i y^{n-i} + y^n \\
&= \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} . \quad \blacksquare
\end{aligned}$$

We now consider the following problem: A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $n_1 + n_2 + \cdots + n_r = n$. How many different divisions are possible? To answer this, we note that there are $\binom{n}{n_1}$ possible choices for the first group; for each choice of the first group there are $\binom{n-n_1}{n_2}$ possible choices for the second group; and so on. Hence, there are $\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_r}{n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}$

possible divisions, denoted by $\binom{n}{n_1, n_2, \ldots, n_r}$.

Example 1-3: In how many ways can a man divide 7 gifts among his 3 children if the eldest is to receive 3 gifts and the others 2 each? **Answer:** $\binom{7}{3,\ 2,\ 2} = \frac{7!}{3!2!2!}$. ◘

Example 1-4: In order to play a game of basketball, 10 boys at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

Solution: Note that this example is different from the previous one since now the order of the two teams is irrelevant. The answer is $\frac{10!}{5!5!}\Big/ 2! = 126$. ◘

<u>The Multinomial Theorem</u>: $(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1, n_2, \ldots, n_r): \\ n_1+n_2+\cdots+n_r=n}} \binom{n}{n_1, n_2, \cdots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$

That is, the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$.

Proof: The proof is left as an exercise. ◘

There are r^n possible outcomes when n ***<u>distinguishable</u>*** balls are to be distributed into r distinguishable urns. Suppose that the n balls are ***<u>indistinguishable</u>*** from each other. In this case, how many different outcomes are possible? In other words, it is to find the number of distinct nonnegative integer-valued vector $(x_1, x_2, \ldots, x_r)$ such that $x_1 + x_2 + \cdots + x_r = n$. It can be imagined that we have n ***indistinguishable*** objects lined up and that we want to divide them into r nonempty groups.

$$\Delta_{\wedge}\Delta_{\wedge}\cdots_{\wedge}\Delta$$

Choose $r-1$ of the space "$_\wedge$" from n objects "Δ."

Figure 1-1

To do so, we can select $r – 1$ of the $n – 1$ spaces between adjacent objects as our dividing points. (See Figure 1-1.) As there are $\binom{n-1}{r-1}$ possible selections, we obtain the following proposition.

<u>Proposition 1-1</u>: There are $\binom{n-1}{r-1}$ distinct **<u>positive</u>** integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying $x_1 + x_2 + \cdots + x_r = n, \quad x_i > 0, i = 1, \ldots, r$. ◘

Note that the number of nonnegative solutions of $x_1 + x_2 + \cdots + x_r = n$ is the same as the number of positive solutions of $y_1 + y_2 + \cdots + y_r = n + r$ (let $y_i = x_i + 1$, $i = 1, \ldots, r$). Hence, from Proposition 1-1, we obtain the following proposition.

***Proposition 1-2*:** There are $\binom{n + r - 1}{r - 1}$ distinct **nonnegative** integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying $x_1 + x_2 + \cdots + x_r = n$. ◘

Example 1-5: An investor has 20 thousand dollars to invest among 4 possible investments. Each investment must be in units of a thousand dollars. If the total 20 thousand is to be invested, how many different investment strategies are possible? What if each investment need be invested at least one thousand dollars? What if not all the money need be invested?

Solution: If we let y_i, i = 1, 2, 3, 4, denote the number of thousands invested in investment number i, then, when all is to be invested, (y_1, y_2, y_3, y_4) are integers satisfying

$$y_1 + y_2 + y_3 + y_4 = 20, \qquad y_i \geq 0.$$

Hence, by Proposition 1-2, there are $\binom{23}{3} = 1771$ possible investment strategies.

If each investment need be invested at least one thousand dollars, if we let x_i, i = 1, 2, 3, 4, be the number of thousands invested in investment i, a strategy is a positive integer-valued vectors (x_1, x_2, x_3, x_4) satisfying

$$x_1 + x_2 + x_3 + x_4 = 20, \qquad x_i > 0.$$

Hence, by Proposition 1-1, there are $\binom{19}{3} = 969$ possible investment strategies.

If not all of the money need be invested, then, if we let y_5 denote the amount kept in reserve, a strategy is a nonnegative inter-valued vector $(y_1, y_2, y_3, y_4, y_5)$ satisfying

$$y_1 + y_2 + y_3 + y_4 + y_5 = 20, \qquad y_i \geq 0.$$

Hence, by Proposition 1-2, there are $\binom{24}{4} = 10626$ possible strategies. ◘

***Proposition 1-3*:** $\binom{n + m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r - 1} + \cdots + \binom{n}{r}\binom{m}{0}$,

whenever $r \leq n,\ r \leq m$.

Proof: Consider a group of n men and m women. We want to find the number of different groups of size r. There are $\binom{n + m}{r}$ groups of size r. As there are $\binom{n}{i}\binom{m}{r - i}$ groups of size r that consist of i men and $r – i$ women, we see that $\binom{n + m}{r} = \sum_{i=0}^{r}\binom{n}{i}\binom{m}{r - i}$. ◘