Chapter V Continuous Distributions

Continuous distributions are generally amenable to more elegant mathematical treatment than are discrete distributions. This makes them especially useful as approximations to discrete distributions. Continuous distributions are used in this way in most applications, both in the construction of models and in applying statistical techniques. An essential property of a continuous random variable is that there is **zero probability** that it takes any specified numerical value, but in general a nonzero probability, calculable as a definite integral of a ***probability density function*** that it takes a value in specified (finite or infinite) intervals.

Some concepts that have great value for discrete distributions are much less valuable in the discussion of continuous distributions, for example, probability generating functions and factorial moments. On the other hand, ***standardization*** (use of the transformed variable to produce a distribution with zero mean and unit standard deviation) is much more useful fro continuous distributions.

5.1 Uniform Distribution

Suppose that a continuous random variable X can assume values only in a bounded interval, say the open interval (a, b), and suppose that the pdf is constant, say $f(x) = c$ over the interval. This implies $c = 1/(b - a)$, since $1 = \int_a^b c\,dx = c(b-a)$. This special distribution is known as the uniform distribution on the interval (a, b), denoted by $X \sim U(a,b)$. The pdf is

$$f(x) = \frac{1}{b-a} \qquad a < x < b$$

and zero otherwise.

The CDF of $X \sim U(a,b)$ has the form

$$F(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a < x < b \\ 1 & b \le x \end{cases}$$

Example 5.1-1: Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits (1) less than 5 minutes for a bus. (2) More than 10 minutes for a bus.

Solution: Let X denote the number of minutes past 7 A.M. that the passenger arrives at the stop. Since X is a uniform random variable over the interval (0, 30), it follows that the passenger will have to wait less than 5 minutes if (and only if) he arrives between 7:10 and 7:15 or between 7:25 and 7:30. Hence the desired probability for (1) is

$$P(10 < X < 15) + P(25 < X < 30) = \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx = \frac{1}{3}.$$

Similarly, he would have to wait more than 10 minutes if he arrives between 7 and 7:05 or between 7:15 and 7:20, and so the probability for (2) is

$$P(0 < X < 5) + P(15 < X < 20) = \int_{0}^{5} \frac{1}{30} dx + \int_{15}^{20} \frac{1}{30} dx = \frac{1}{3}. \quad \square$$

Theorem 5.1-1: If $X \sim \mathrm{U}(a, b)$, then

$$\mu = \frac{a+b}{2}, \qquad \sigma^2 = \frac{(b-a)^2}{12}, \qquad \text{and} \qquad M(t) = \begin{cases} \dfrac{e^{tb} - e^{ta}}{t(b-a)}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Proof: $E(X) = \int_a^b x f(x)\, dx = \int_a^b \frac{x}{b-a} dx = \left. \frac{x^2}{2(b-a)} \right|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}.$

$$E(X^2) = \int_a^b \frac{x^2}{b-a} dx = \left. \frac{x^3}{3(b-a)} \right|_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + ab + a^2}{3}.$$

$$\sigma^2 = E\left(X^2\right) - [E(X)]^2 = \frac{b^2 + ab + a^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{(b-a)^2}{12}.$$

$$M(t) = E\left(e^{tX}\right) = \int_a^b \frac{e^{tx}}{b-a} dx = \left. \frac{e^{tx}}{t(b-a)} \right|_a^b = \frac{e^{tb} - e^{ta}}{t(b-a)}, \qquad \text{for } t \neq 0. \quad \square$$

Perhaps a more important application of the uniform distribution occurs in the case of computer simulation, which relies on the generation of "random number." Random number generators are functions in the computer language, or in some cases subroutines in programs, which are designed to produce numbers that behave as if they were data from $U(0,1)$ with CDF $F(x) = x$ for $0 < x < 1$. We will discuss this in Chapter VII.

5.2 Exponential Distribution

When previously observing a process of the (approximate) Poisson type, we counted the number of changes occurring in a given interval. This number was a discrete-type random variable with a Poisson distribution. But not only is the number of changes a random variable; the waiting times between successive changes are also random variables. However, the latter are of the continuous type, since each of them can assume any positive value. In particular, let W denote the waiting time until the first change occurs when observing a Poisson process in which the mean number of changes in the unit interval is λ. Then W is a continuous-type random variable, and we proceed to find its distribution function.

Clearly, this waiting time is nonnegative; thus the distribution function $F(w) = 0$, $w < 0$. For $w \ge 0$,

$$F(w) = P(W \le w) = 1 - P(W > w) = 1 - P(\text{no chnages in } [0,\text{w}]) = 1 - e^{-\lambda w},$$

since we previously discovered that $e^{-\lambda w}$ equals the probability of no changes in an interval of length w. Thus, when $w > 0$, the pdf of W is given by

$$F'(w) = \lambda e^{-\lambda w} = f(w).$$

We often let $\lambda = 1/\theta$ and say that the random variable X has an ***exponential distribution***, denoted by $X \sim \text{EXP}(\theta)$, if its pdf is defined by

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \qquad 0 \le x < \infty,$$

where the parameter $\theta > 0$. Its CDF is $F(x) = 1 - \exp(-x/\theta)$, $x \ge 0$.

It is easy to be verified that $\int_0^\infty \frac{1}{\theta} \exp\left(\frac{-x}{\theta}\right) dx = 1$.

Theorem 5.2-1: If $X \sim \text{EXP}(\theta)$, then

$$\mu = \theta, \qquad \sigma^2 = \theta^2, \qquad \text{and} \qquad M(t) = \frac{1}{1-\theta t}, \quad t < \frac{1}{\theta}.$$

Proof:

$$M(t) = \int_0^\infty e^{tx} \frac{1}{\theta} \exp\left(\frac{-x}{\theta}\right) dx = \int_0^\infty \frac{1}{\theta} \exp\left(\frac{-(1-\theta t)x}{\theta}\right) dx$$

$$= \frac{1}{1-\theta t} \int_0^\infty \frac{1}{\theta/(1-\theta t)} \exp\left(\frac{-x}{\theta/(1-\theta t)}\right) dx = \frac{1}{1-\theta t}, \quad t < \frac{1}{\theta}.$$

Let $R(t) = \ln M(t) = -\ln(1-\theta t)$. Then

$$\mu = R'(t)\Big|_{t=0} = \left.\frac{-(-\theta)}{1-\theta t}\right|_{t=0} = \theta.$$

$$\sigma^2 = R''(t)\Big|_{t=0} = \left.\frac{-\theta(-\theta)}{(1-\theta t)^2}\right|_{t=0} = \theta^2. \quad \blacksquare$$

According to the Theorem 5.2-1, if λ is the mean number of changes in the unit interval, then $\theta = 1/\lambda$ is the mean waiting time for the first change.

Example 5.2-1: Customers arrive in a certain shop according to an approximate Poisson process at a mean rate of 20 per hour. What is the probability that the shopkeeper will have to wait more than 5 minutes for the arrival of the first customer?

Solution: Let X denote the waiting time in ***minutes*** until the first customer arrives and note that $\lambda = 1/3$ is the expected number of arrives per minutes. Thus

$$\theta = \frac{1}{\lambda} = 3$$

and

$$f(x) = \frac{1}{3}\exp\left(\frac{-x}{3}\right), \qquad 0 \le x < \infty.$$

Hence,

$$P(X > 5) = \int_5^\infty \frac{1}{3}\exp\left(\frac{-x}{3}\right)dx = \exp\left(\frac{-5}{3}\right) = 0.189.$$

The median time until the first arrival is

$$m = -3\ln(0.5) = 2.079. \quad \blacksquare$$

Theorem 5.2-2 (memoryless property):

If $X \sim \text{EXP}(\theta)$, then $P\big(X > a + t \mid X > a\big) = P(X > t)$ for all $a > 0$ and $t > 0$.

Proof: $$P(X > a + t \mid t > a) = \frac{P(X > a + t \text{ and } X > a)}{P(X > a)} = \frac{P(X > a + t)}{P(X > a)}$$

$$= \frac{\exp\big(-(a+t)/\theta\big)}{\exp\big(-a/\theta\big)} = P(X > t). \quad \blacksquare$$

5.3 Gamma and Chi-Square Distributions

In the (approximate) Poisson process with mean λ, we have seen that the waiting time until the first change has an exponential distribution. We now let W denote the waiting time until the αth change occurs and find the distribution of W.

The distribution function of W, when $w > 0$, is given by

$$F(w) = P(W \le w) = 1 - P(W > w) = 1 - P(\text{fewer than } \alpha \text{ changes occur in}[0, \text{w}])$$

$$= 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{-\lambda w}}{k!}, \qquad (5.3.1)$$

since the number of changes in the interval [0, w] has a Poisson distribution with mean λw. Because W is a continuous-type random variable, $F'(w)$ is equal to the pdf of W whenever this derivative exists. We have, provided $w > 0$, that

$$F'(w) = \lambda e^{-\lambda w} - e^{-\lambda w} \sum_{k=1}^{\alpha-1} \left[\frac{k(\lambda w)^{k-1}\lambda}{k!} - \frac{(\lambda w)^k \lambda}{k!} \right] = \lambda e^{-\lambda w} - e^{-\lambda w} \sum_{k=1}^{\alpha-1} \left[\frac{(\lambda w)^{k-1}\lambda}{(k-1)!} - \frac{(\lambda w)^k \lambda}{k!} \right]$$

$$= \lambda e^{-\lambda w} - e^{-\lambda w} \left[\lambda - \frac{(\lambda w)^{\alpha-1}\lambda}{(\alpha-1)!} \right] = \frac{\lambda^\alpha w^{\alpha-1}}{(\alpha-1)!} e^{-\lambda w}.$$

Of course, if $w < 0$, then $F(w) = 0$ and $F'(w) = 0$. A pdf of this form is said to be one of the gamma type, and the random variable W is said to have a **gamma distribution.**

Before determining the characteristics of the gamma distribution, let us consider the gamma function for which the distribution is named. The ***gamma function*** is defined by

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx, \qquad t > 0.$$

This integral is obviously positive for $t > 0$. If $t > 1$, integration of the gamma function of t by parts yields

$$\Gamma(t) = \left(-x^{t-1}e^{-x}\right)\Big|_0^\infty + \int_0^\infty (t-1)x^{t-2}e^{-x}dx = (t-1)\int_0^\infty x^{t-2}e^{-x}dx$$

$$= (t-1)\Gamma(t-1).$$

For example, $\Gamma(6) = 5\Gamma(5)$ and $\Gamma(3.5) = (2.5)\Gamma(2.5)$. Whenever, $t = n$, a positive integer, we have, by repeated application of $\Gamma(t) = (t-1)\Gamma(t-1)$, that

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\cdots(2)(1)\Gamma(1).$$

However,

$$\Gamma(1) = \int_0^\infty e^{-x} dx = 1.$$

Thus, when n is a positive integer, we have that

$$\Gamma(n) = (n-1)!.$$

Let us now formally define the pdf of the gamma distribution and find its characteristics.

Definition 5.3-1: A random variable X is said to have a ***gamma distribution*** with ***parameters*** $\alpha > 0$ and $\theta > 0$, denoted by $X \sim G(\alpha,\theta)$, if and only if the density function of X is

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1}e^{-x/\theta}, \qquad 0 \le x < \infty. \quad \square$$

According to the **Definition 5.3-1**, W, the waiting time until the αth change in a Poisson process, has a gamma distribution with parameters α and $\theta = 1/\lambda$. By the change of variables $y = x/\theta$, it can be shown that

$$\int_0^{\infty} \frac{1}{\theta^{\alpha}} x^{\alpha-1}e^{-x/\theta}dx = \int_0^{\infty} y^{\alpha-1}e^{-y}dy = \Gamma(\alpha).$$

Hence,
$$\int_0^{\infty} f(x)dx = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1}e^{-x/\theta}dx = \frac{1}{\Gamma(\alpha)}\Gamma(\alpha) = 1.$$

Example 5.3-1: Suppose that an average of 30 customers per hour arrive at a shop in accordance with a Poisson process. That is, if a minute is our unit, then $\lambda = 1/2$. What is the probability that the shopkeeper will wait more than 5 minutes before both of the first two customers arrive?

Solution: If X denotes the waiting time in minutes until the second customer arrives, then X has a gamma distribution with $\alpha = 2$ and $\theta = 1/\lambda = 2$. Hence,

$$P(X > 5) = \int_5^{\infty} \frac{1}{\Gamma(2)2^2} x^{2-1}e^{-x/2}dx = \int_5^{\infty} \frac{1}{4} xe^{-x/2}dx = \frac{1}{4}\Big[(-2)xe^{-x/2} - 4e^{-x/2}\Big]_5^{\infty}$$

$$= \frac{7}{2}e^{-5/2} = 0.287.$$

We would also have used equation (5.3.1) with $\lambda = 1/\theta$ because α is an integer. From equation (5.3.1) we have

$$P(X > 5) = \sum_{k=0}^{\alpha-1} \frac{(x/\theta)^k e^{-x/\theta}}{k!} = \sum_{k=0}^{2-1} \frac{(5/2)^k e^{-5/2}}{k!} = e^{-5/2}\left(1 + \frac{5}{2}\right) = \frac{7}{2}e^{-5/2}. \quad \square$$

Theorem 5.3-1: If $X \sim G(\alpha,\theta)$, then

$$\mu = \alpha\theta, \qquad \sigma^2 = \alpha\theta^2, \qquad \text{and} \qquad M(t) = (1-\theta t)^{-\alpha} \qquad t < 1/\theta.$$

Proof: $E(X) = \int_0^\infty x \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta} dx = \frac{\Gamma(\alpha+1)\theta^{\alpha+1}}{\Gamma(\alpha)\theta^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha+1)\theta^{\alpha+1}} x^{(\alpha+1)-1} e^{-x/\theta} dx$

$$= \frac{\alpha\Gamma(\alpha)\theta\theta^\alpha}{\Gamma(\alpha)\theta^\alpha} = \alpha\theta .$$

$$E(X^2) = \int_0^\infty x^2 \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta} dx = \frac{\Gamma(\alpha+2)\theta^{\alpha+2}}{\Gamma(\alpha)\theta^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha+2)\theta^{\alpha+2}} x^{(\alpha+2)-1} e^{-x/\theta} dx$$

$$= \frac{(\alpha+1)\alpha\Gamma(\alpha)\theta^2\theta^\alpha}{\Gamma(\alpha)\theta^\alpha} = (\alpha+1)\alpha\theta^2 .$$

$$\sigma^2 = E(X^2) - [E(X)]^2 = (\alpha+1)\alpha\theta^2 - \alpha^2\theta^2 = \alpha\theta^2 .$$

$$M(t) = \int_0^\infty \frac{e^{tx}}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} \exp\left(\frac{-x}{\theta}\right) dx$$

$$= \frac{[\theta/(1-\theta t)]^\alpha}{\theta^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)[\theta/(1-\theta t)]^\alpha} x^{\alpha-1} \exp\left\{\frac{-x}{\theta/(1-\theta t)}\right\} dx$$

$$= (1-\theta t)^{-\alpha} . \quad \blacksquare$$

One of the important properties of the gamma distribution (including the exponential, chi-square distributions, and so on), is its ***<u>reproductive property</u>***: If X_1 and X_2 are independently distributed as $G(\alpha_1, \theta)$ and $G(\alpha_2, \theta)$, respectively, then $(X_1 + X_2) \sim G((\alpha_1 + \alpha_2), \theta)$. We will discuss this in Chapter VII.

Definition 5.3-2: The gamma distribution with $\theta = 2$ and $\alpha = r/2$ (r being a positive integer) is called the χ_r^2 (read "**chi-square**") distribution with r degrees of freedom. $\blacksquare$

Note that for $r \le 2$ the mode of the chi-square distribution is at 0 while for $r > 2$ the mode is at $r - 2$.

Theorem 5.3-2: If $X \sim \chi_r^2$, then

$$\mu = \alpha\beta = r, \qquad \sigma^2 = \alpha\beta^2 = 2r, \qquad \text{and} \qquad M(t) = (1-2t)^{-r/2} \qquad t < 1/2 . \quad \blacksquare$$

5.4 Normal Distribution

The ***normal distribution*** (or called the ***Gaussian distribution***) is perhaps the most important distribution in statistical applications since many measurements have (approximate) normal distributions. It was introduced by the French mathematician Abrahan de Moivre in 1733 and was used by him to approximate probabilities associated with binomial random variables when the binomial parameter n is **large.** This result was later extended by Laplace and others and is now encompassed in a probability theorem known as the ***central limit theorem***. The central limit theorem, one of the two most important results in probability theory (the other being the strong law of large number), gives a theoretical base to the often noted empirical observation that, in practice, many random phenomena obey, at least approximately, a normal probability distribution. We will discuss one form of this theorem in Chapter VIII.

We say that X is a normal random variable, or simply that X is normally distributed, with **parameters** μ and σ^2 denoted by $X \sim N(\mu, \sigma^2)$ if the density of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \qquad -\infty < x < \infty .$$

This density function is a **bell-shaped curve** that is **symmetric** about μ. The values μ and σ^2, satisfying $-\infty < \mu < \infty$ and $0 < \sigma^2 < \infty$, represent, in some sense, the average value and the possible variation of X.

To prove that $f(x)$ is indeed a probability density function, we need to show that

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-(x-\mu)^2/2\sigma^2\right\} dx = 1 .$$

By making the substitution $y = (x-\mu)/\sigma$, we see that

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-(x-\mu)^2/2\sigma^2\right\} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-y^2/2\right\} dy$$

and hence we must show that

$$\int_{-\infty}^{\infty} \exp\left\{-y^2/2\right\} dy = \sqrt{2\pi} .$$

Let $I = \int_{-\infty}^{\infty} \exp\left\{-y^2/2\right\} dy$. Since $I > 0$, if $I^2 = 2\pi$, then $I = \sqrt{2\pi}$. Now

$$I^2 = \int_{-\infty}^{\infty} \exp\left\{-y^2/2\right\} dy \int_{-\infty}^{\infty} \exp\left\{-x^2/2\right\} dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\left(y^2 + x^2\right)/2\right\} dy dx .$$

We now evaluate the double integral by means of a change of variables to polar coordinates. That is, $x = r\cos\theta$, $y = r\sin\theta$, $dydx = r\,d\theta\,dr$. Thus

$$I^2 = \int_0^\infty \int_0^{2\pi} e^{-r^2/2} r\, d\theta\, dr = 2\pi \int_0^\infty r e^{-r^2/2}\, dr = -2\pi e^{-r^2/2}\Big|_0^\infty = 2\pi .$$

Hence $I = \sqrt{2\pi}$, and the result is proved.

Theorem 5.4-1: If $X \sim N(\mu, \sigma^2)$, then

$$E(X) = \mu, \qquad \mathrm{Var}(X) = \sigma^2, \qquad \text{and} \qquad M(t) = \exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}.$$

Proof: $E(X) = \displaystyle\int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \underbrace{=}_{\text{Let } z=(x-\mu)/\sigma} \int_{-\infty}^{\infty} \frac{(\sigma z + \mu)}{\sigma\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right)\sigma\, dz$

$$= \sigma \int_{-\infty}^{\infty} \frac{z}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz$$

$$= \sigma \left[\frac{-1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right)\right]\Bigg|_{-\infty}^{\infty} + \mu = \mu .$$

$$\mathrm{Var}(X) = E\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

$$\underbrace{=}_{\text{Let } z=(x-\mu)/\sigma} \int_{-\infty}^{\infty} \frac{\sigma^2 z^2}{\sigma\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right)\sigma\, dz .$$

Let $u = z$ and $z \exp\left(-z^2/2\right)dz = dv$. Then, $du = dz$, and $-\exp\left(-z^2/2\right) = v$. The integral by parts yields

$$\mathrm{Var}(X) = \frac{\sigma^2}{\sqrt{2\pi}} \left[\left(-z \exp\left(\frac{-z^2}{2}\right)\right)\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \exp\left(\frac{-z^2}{2}\right) dz\right] = \sigma^2 .$$

$$M(t) = \int_{-\infty}^{\infty} \frac{e^{tx}}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}\left[x^2 - 2\left(\mu + \sigma^2 t\right)x + \mu^2\right]\right\} dx .$$

To evaluate this integral, we complete the square in the exponent

$$x^2 - 2\left(\mu + \sigma^2 t\right)x + \mu^2 = \left[x - \left(\mu + \sigma^2 t\right)\right]^2 - 2\mu\sigma^2 t - \sigma^4 t^2 .$$

Thus, $\displaystyle M(t) = \exp\left(\frac{2\mu\sigma^2 t + \sigma^4 t^2}{2\sigma^2}\right) \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}\left[x - \left(\mu + \sigma^2 t\right)\right]^2\right\} dx .$

Note that the integrand in the last integral is like the pdf of a normal distribution with μ replaced by $\mu + \sigma^2 t$. However, the normal pdf integrates to one for all real μ, in particular when it equals $\mu + \sigma^2 t$. Thus

$$M(t) = \exp\left(\frac{2\mu\sigma^2 t + \sigma^4 t^2}{2\sigma^2}\right) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right). \quad \square$$

Definition 5.4-1: If Z is $N(0,1)$, we say that Z has a standard normal distribution. Its pdf denoted by $\phi(z)$ and CDF denoted by $\Phi(z)$ are

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$$

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt = \int_{-\infty}^{z} \phi(t)\, dt. \quad \square$$

It is not possible to evaluate the integral $\Phi(z)$ by finding an anti-derivative that can be expressed as an elementary function. However, numerical approximations for integrals of this type have been tabulated. Because of the symmetry of the standard normal pdf, we have that $\Phi(-z) = 1 - \Phi(z)$ and $\phi(-z) = \phi(z)$ for all z. Furthermore, due to the special form of $\phi(z)$, we have

$$\phi'(z) = -z\phi(z) \qquad \text{and} \qquad \phi''(z) = \left(z^2 - 1\right)\phi(z).$$

Consequently, $\phi(z)$ has a unique maximum at $z = 0$ and inflection points at $z = \pm 1$. Note also that $\phi(z) \to 0$ and $\phi'(z) \to 0$ as $z \to \pm\infty$.

Theorem 5.4-2: If X is $N(\mu, \sigma^2)$, then $Z = (X - \mu)/\sigma$ is $N(0,1)$.

Proof: The distribution function of Z is

$$P(Z \le z) = P\left(\frac{X - \mu}{\sigma} \le z\right) = P\left(X \le z\sigma + \mu\right) = \int_{-\infty}^{z\sigma+\mu} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

$$\underbrace{=}_{\text{Let } t=(x-\mu)/\sigma} \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right] dt.$$

But this is the expression for $\Phi(z)$, the distribution function of a standardized normal random variable. Hence Z is $N(0,1)$. $\square$

Theorem 5.4-3: If $Z \sim N(0,1)$, then $Y = Z^2 \sim \chi_1^2$.

Proof: The distribution function $F(y)$ of Y is, for $y \ge 0$,

$$F(y) = P(Y \le y) = P(Z^2 \le y) = P(-\sqrt{y} \le Z \le \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz = 2\int_{0}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz .$$

If we change the variable of integration by writing $z = \sqrt{x}$, then we have

$$F(y) = \int_0^y \frac{1}{\sqrt{2\pi x}} \exp\left(\frac{-x}{2}\right) dx, \qquad y \ge 0.$$

Of course, $F(y) = 0$ for $y < 0$. Hence the pdf $f(y) = F'(y)$ of the continuous-type random variable Y is, by one form of the fundamental theorem of calculus,

$$f(y) = \frac{1}{\sqrt{\pi}\sqrt{2}} y^{(1/2)-1} \exp\left(\frac{-y}{2}\right), \qquad y \ge 0.$$

Since $$\Gamma(1/2) = \int_0^\infty x^{(1/2)-1} \exp(-x) dx$$

$$\underbrace{=}_{\text{Let } y=\sqrt{2x}} \int_0^\infty \sqrt{2} \exp\left(\frac{-y^2}{2}\right) dy$$

$$= 2\sqrt{\pi} \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y^2}{2}\right) dy$$

$$= 2\sqrt{\pi}\,\frac{1}{2} = \sqrt{\pi},$$

$$f(y) = \frac{1}{\Gamma(1/2)\, 2^{1/2}} y^{(1/2)-1} \exp\left(\frac{-y}{2}\right), \qquad y \ge 0.$$

That is, $Y = Z^2 \sim \chi_1^2$. $\square$

5.5 Beta Distribution

A random variable X is said to have a **beta distribution** with the parameters $\alpha > 0$ and $\beta > 0$, denoted $X \sim \text{BETA}(\alpha, \beta)$ if its density is given by

$$f(x) = \begin{cases} \dfrac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Theorem 5.5-1: $\displaystyle\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Proof: $\displaystyle\Gamma(\alpha)\Gamma(\beta) = \left(\int_0^\infty x^{\alpha-1}e^{-x}dx\right)\left(\int_0^\infty y^{\beta-1}e^{-y}dy\right) = \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}e^{-(x+y)}dx\,dy$.

Let $u = \dfrac{x}{x+y}$, so that $x = \dfrac{uy}{1-u}$, $dx = \dfrac{y\,du}{(1-u)^2}$, $u \in (0,1)$ and $x + y = \dfrac{y}{1-u}$.

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty\int_0^1 \frac{u^{\alpha-1}}{(1-u)^{\alpha-1}} y^{\alpha-1}y^{\beta-1}e^{-y/(1-u)} y \frac{du}{(1-u)^2} dy$$

$$= \int_0^\infty\int_0^1 \frac{u^{\alpha-1}}{(1-u)^{\alpha+1}} y^{\alpha+\beta-1}e^{-y/(1-u)} du\,dy.$$

Let $y/(1-u) = v$, so that $y = v(1-u)$, $dy = (1-u)dv$, $v \in (0,\infty)$. Then the integral is

$$= \int_0^\infty\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}v^{\alpha+\beta-1}e^{-v} du\,dv$$

$$= \int_0^\infty v^{\alpha+\beta-1}e^{-v}dv \int_0^\infty u^{\alpha-1}(1-u)^{\beta-1} du$$

$$= \Gamma(\alpha+\beta)\int_0^\infty u^{\alpha-1}(1-u)^{\beta-1} du. \quad \square$$

According to the Theorem 5.5-1, $\displaystyle\int_0^1 f(x)dx = 1$.

Remark: For $\alpha = \beta = 1$, we get the $U(0,1)$, since $\Gamma(1) = 1$ and $\Gamma(2) = 1 \times \Gamma(1) = 1$.

The beta distribution is often used as a model for ***proportions***, such as the proportion of impurities in a chemical product or the proportion of time that a machine is under repair. It can also be used to model a random phenomenon whose set of possible values is some ***finite*** interval [a,

b] — which by letting a denote the origin and taking (b − a) as a unit measurement can be transformed into the interval [0, 1].

When $\alpha = \beta$, the beta density is symmetric about 1/2, giving more and more weight to region about 1/2 as the common value α increases. When $\beta > \alpha$, the density is skewed to the left (in the sense that smaller values become more likely); and it is skewed to the right when $\alpha > \beta$.

Theorem 5.5-2: If $X \sim \text{BETA}(\alpha, \beta)$, then

$$\mu = \frac{\alpha}{\alpha+\beta} \qquad \text{and} \qquad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}.$$

Proof: Left as an excise. ◘

Example 5.5-1: A gasoline wholesale distributor has bulk storage tanks that hold fixed supplies and are filled every Monday. Of interest to the wholesaler is the proportion of this supply that is sold during the week. Over many weeks of observations, the distributor found that this proportion could be modeled by a beta distribution with $\alpha = 4$ and $\beta = 2$. Find the probability that the wholesaler will sell at least 90% of her stock in a given week.

Solution: If X denote the proportion sold during the week, then

$$f(x) = \frac{\Gamma(4+2)}{\Gamma(4)\Gamma(2)} x^3(1-x), \qquad 0 \le x \le 1$$

and

$$P(x > 0.9) = \int_{0.9}^{1} 20\left(x^3 - x^4\right)dx = 20\left\{\left.\frac{x^4}{4}\right]_{0.9}^{1} - \left.\frac{x^5}{5}\right]_{0.9}^{1}\right\} = 20(0.004) = 0.08$$

It is not very likely that 90% of the stock will be sold in a given week. ◘

5.6 Mixed Distribution

It is possible to have a random variable whose distribution is neither purely discrete nor continuous. A probability distribution for a random variable X is of mixed type if the CDF has the form

$$F(x) = aF_d(x) + (1-a)F_c(x)$$

where $F_d(x)$ and $F_c(x)$ are CDFs of discete and continuous type, respectively, and $0 < a < 1$.

Example 5.6-1: Suppose that a driver encounters a stop sign and either waits for a random period of time before proceeding or proceeds immediately. An appropriate model would allow the waiting time to be either zero or positive, both with nonzero probability. Let the CDF of the waiting time X be

$$F(x) = 0.4F_d(x) + 0.6F_c(x) = 0.4 + 0.6(1 - e^{-x})$$

where $F_d(x) = 1$ and $F_c(x) = 1 - e^{-x}$ if $x \geq 0$, and both are zero if $x < 0$. The probability of proceeding immediately is $P(X = 0) = 0.4$. The probability that the waiting time is less than 0.5 minutes is

$$P(X \leq 0.5) = 0.4 + 0.6\left(1 - e^{-0.5}\right) = 0.636\,.$$

The distribution of X given $X > 0$ corresponds to

$$P(X \leq x \mid X > 0) = \frac{P(X > 0 \text{ and } X \leq x)}{P(X > 0)} = \frac{P(0 < X \leq x)}{P(X > 0)} = \frac{F(x) - F(0)}{1 - F(0)}$$

$$= \frac{0.4 + 0.6\left(1 - e^{-x}\right) - 0.4}{1 - 0.4} = 1 - e^{-x}\,. \quad \blacksquare$$

Example 5.6-2: The distribution function of the random variable Y is given by

$$F(y) = \begin{cases} 0 & y < 0 \\ y^2/4 & 0 \leq y < 1 \\ 1/2 & 1 \leq y < 2 \\ y/3 & 2 \leq y < 3 \\ 1 & 3 \leq y. \end{cases}$$

A graph of $F(y)$ is presented in Figure 5.6-1. Probabilities can be computed using $F(y)$:

$$P(0 < Y < 1) = 1/4$$

$$P(0 < Y \leq 1) = 1/2$$

$$P(Y = 1) = 1/2$$

$$P(1 \le Y \le 2) = 2/3 - 1/4 = 5/12.$$

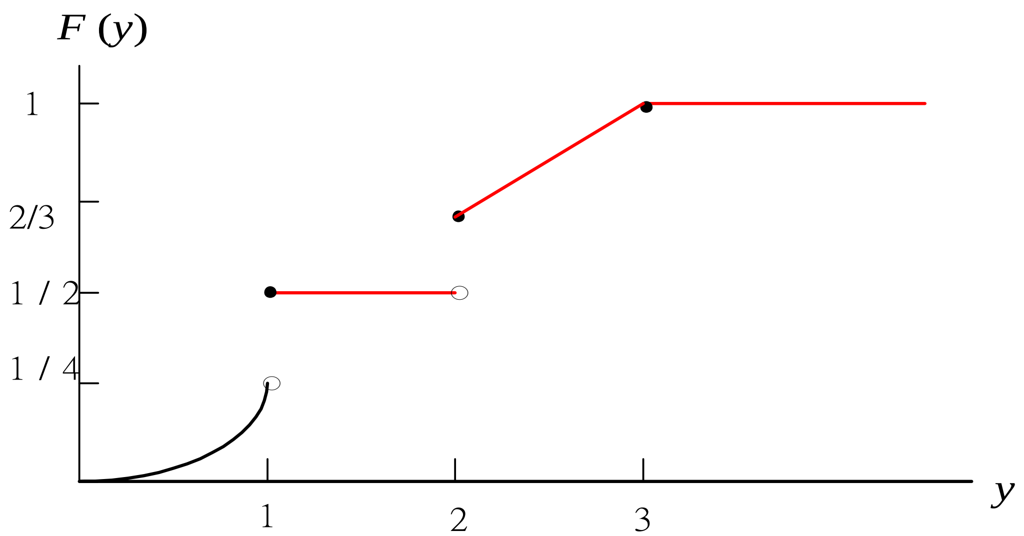

Figure 5.6-1 Graph of $F(y)$

We now find the mean and variance for the random variable Y. **Note** that there $F'(y) = y/2$ when $0 < y < 1$, and $F'(y) = 1/3$ when $2 < y < 3$; also $P(Y = 1) = 1/4$ and $P(Y = 2) = 1/6$. Accordingly, we have

$$\mu = E(Y) = \int_0^1 y(y/2)dy + 1(1/4) + 2(1/6) + \int_2^3 y(1/3)dy = 19/12.$$

$$\sigma^2 = E(Y^2) - [E(Y)]^2 = \int_0^1 y^2(y/2)dy + 1^2(1/4) + 2^2(1/6) + \int_2^3 y^2(1/3)dy - (19/12)^2$$

$$= 1/48. \quad \blacksquare$$