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9.2 Efficiency (Rao-Cramer Inequality)

It is usually possible to obtain more than one unbiased estimator for the same target parameter θ. If $\hat{\theta}_1$ and $\hat{\theta}_2$ denote two unbiased estimators for the same parameter θ, we prefer to use the estimator with the smaller variance. That is, if both estimators are unbiased, $\hat{\theta}_1$ is relatively more efficient than $\hat{\theta}_2$ if $Var(\hat{\theta}_2) > Var(\hat{\theta}_1)$.

Definition 9.2-1: The **relative efficiency** of an unbiased estimator $\hat{\theta}$ of θ to another unbiased estimator $\hat{\theta}^*$ of θ is given by

$$\mathrm{Re}(\hat{\theta},\hat{\theta}^*) = \frac{Var(\hat{\theta}^*)}{Var(\hat{\theta})}.$$

An unbiased estimator $\hat{\theta}^*$ of θ is said to be **efficient** if $\mathrm{Re}(\hat{\theta},\hat{\theta}^*) \le 1$ for all unbiased estimator $\hat{\theta}$ of θ, and all $\theta \in \mathbf{\Omega}$. The **efficiency** of an unbiased estimator $\hat{\theta}$ of θ t is given by

$$e(\hat{\theta},\hat{\theta}^*) = \mathrm{Re}(\hat{\theta},\hat{\theta}^*)$$

if $\hat{\theta}^*$ is an efficient estimator of θ. ◘

An interesting result by Rao-Cramer helps decide among several estimators since it provides a lower bound for the variance of every **unbiased** estimator of θ. Thus we know that if a certain unbiased estimator has a variance equal to that lower bound, we cannot find a better one and hence it is the best in the sense of being the **uniformly minimum variance unbiased estimator** (UMVUE). We describe the **Cramer-Rao lower bound** (**CRLB**) here without proof.

Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution with p.d.f. $f(x;\theta)$, $\theta \in \mathbf{\Omega} = \{\theta : c < \theta < d\}$, where the support of X does not depend on θ so that we can differentiate, with respect to θ, under integral signs like that in the following integral:

$$\int_{-\infty}^{\infty} f(x;\theta)\, d\theta = 1.$$

If $Y = u(X_1, X_2, \cdots, X_n)$ is an **unbiased** estimator of θ, then

$$Var(Y) \ge \frac{1}{n\int_{-\infty}^{\infty} \{[\partial \ln f(x;\theta)/\partial\theta]\}^2 f(x;\theta)\, dx}$$

$$= \frac{1}{-n\int_{-\infty}^{\infty} \{[\partial^2 \ln f(x;\theta)/\partial\theta^2]\} f(x;\theta)\, dx}$$

Note that the two integrals in the respective denominators are the expectations

$$E\left\{\left[\frac{\partial \ln f(x;\theta)}{\partial\theta}\right]^2\right\} \quad \text{and} \quad E\left\{\left[\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right]\right\};$$

sometimes one is easier to compute than the other. It can be shown that

$$E\left\{\left[\frac{\partial \ln f(x;\theta)}{\partial\theta}\right]^2\right\} = -E\left\{\left[\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right]\right\}$$

The CRLB can be discussed more generally as follows: If $Y = u(X_1, X_2, \cdots, X_n)$ is an **unbiased** estimator of $\tau(\theta)$, then

$$Var(Y) \geq \frac{[\tau'(\theta)]^2}{n\int_{-\infty}^{\infty}\{[\partial \ln f(x;\theta)/\partial\theta]\}^2 f(x;\theta)\,dx}$$

$$= \frac{[\tau'(\theta)]^2}{-n\int_{-\infty}^{\infty}\{[\partial^2 \ln f(x;\theta)/\partial\theta^2]\} f(x;\theta)\,dx}.$$

Example 9.2-2: Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from $N(\mu, \sigma^2)$. Assume that σ^2 is known and set $\mu = \theta$. Then

$$f(x;\theta) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[\frac{-(x-\theta)^2}{2\sigma^2}\right], \qquad -\infty < x < \infty$$

and hence

$$\ln f(x;\theta) = \ln\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right) - \frac{(x-\theta)^2}{2\sigma^2}.$$

Next,

$$\frac{\partial}{\partial\theta}\ln f(x;\theta) = \frac{1}{\sigma}\frac{(x-\theta)}{\sigma},$$

so that

$$\left[\frac{\partial}{\partial\theta}\ln f(x;\theta)\right]^2 = \frac{1}{\sigma^2}\frac{(x-\theta)^2}{\sigma^2}.$$

Then

$$E\left[\frac{\partial}{\partial\theta}\ln f(X;\theta)\right]^2 = \frac{1}{\sigma^2}.$$

Thus the CRLB is σ^2/n. Once again, $\overline{X}$ is an unbiased estimator of θ and its variance is equal to σ^2/n, that is the CRLB. Therefore, $\overline{X}$ is a UMVUE of θ.

Note that X_1 is another unbiased estimator of θ with variance σ^2. Its **efficiency** is $(\sigma^2/n)/\sigma^2 = 1/n$. This is the one of reasons to choose $\overline{X}$ to estimate θ instead of any individual observation. ◘

Example 9.2-3: Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from $N(\mu, \sigma^2)$. Suppose that μ is known and set $\sigma^2 = \theta$. Then

$$f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}}\exp\left[\frac{-(x-\mu)^2}{2\theta}\right], \qquad -\infty < x < \infty,$$

so that

$$\ln f(x;\theta) = \frac{-1}{2}\ln(2\pi) - \frac{1}{2}\ln\theta - \frac{(x-\mu)^2}{2\theta}$$

and

$$\frac{\partial}{\partial\theta}\ln f(x;\theta) = \frac{-1}{2\theta} + \frac{(x-\mu)^2}{2\theta^2}.$$

Then

$$\left[\frac{\partial}{\partial\theta}\ln f(x;\theta)\right]^2 = \frac{1}{4\theta^2} - \frac{1}{2\theta^2}\left(\frac{x-\mu}{\sqrt{\theta}}\right)^2 + \frac{1}{4\theta^2}\left(\frac{x-\mu}{\sqrt{\theta}}\right)^4$$

and since $(X-\mu)/\sqrt{\theta}$ is $N(0, 1)$, we obtain

$$E\left(\frac{x-\mu}{\sqrt{\theta}}\right)^2 = 1 \quad \text{and} \quad E\left(\frac{x-\mu}{\sqrt{\theta}}\right)^4 = 3.$$

Therefore

$$E\left[\frac{\partial}{\partial\theta}\ln f(X;\theta)\right]^2 = \frac{1}{2\theta^2}$$

and the CRLB is $2\theta^2/n$. Next,

$$\sum_{j=1}^{n}\left(\frac{X_j-\mu}{\sqrt{\theta}}\right)^2 \sim \chi^2(n),$$

so that

$$E\left[\sum_{j=1}^{n}\left(\frac{X_j-\mu}{\sqrt{\theta}}\right)^2\right] = n \qquad \text{and} \quad Var\left[\sum_{j=1}^{n}\left(\frac{X_j-\mu}{\sqrt{\theta}}\right)^2\right] = 2n.$$

Therefore $(1/n)\sum_{j=1}^{n}(X_j-\mu)^2$ is an unbiased estimator of θ and its variance is $2\theta^2/n$, equal to the CRLB. Thus $(1/n)\sum_{j=1}^{n}(X_j-\mu)^2$ is UMVUE of θ. ◘

Example 9.2-4: Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from $N(\mu, \sigma^2)$. We assume that both μ and σ^2 are unknown and set $\mu = \theta_1$ and $\sigma^2 = \theta_2$. Suppose that we are interested in finding a UMVUE estimator of θ_2. It can be shown that the CRLB is again equal to $2\theta_2^2/n$ by treating θ_1 as a constant and θ_2 as the parameter θ. We have shown that $S^2 = \sum_{j=1}^{n}(X_j - \overline{X})^2/(n-1)$ is an unbiased estimator of θ_2. Since

$$\frac{(n-1)S^2}{\theta_2} = \sum_{j=1}^{n}\left(\frac{X_j-\overline{X}}{\sqrt{\theta_2}}\right)^2 \sim \chi^2(n-1),$$

it follows that

$$Var(S^2) = Var\left[\frac{1}{n-1}\sum_{j=1}^{n}(X_i-\overline{X})^2\right] = \frac{2\theta_2^2}{n-1} > \frac{2\theta_2^2}{n},$$

the RCLB. However, $S^2 = \sum_{j=1}^{n} (X_j - \mu)^2 \big/ (n-1)$ is indeed a UMVUE of θ_2 (it will be shown later). That is, the CRLB is not attained for the UMVUE estimator of θ_2, when θ_1 is unknown. ◘