國立高雄大學九十六學年度博士班招生考試試題

科目：機率論 系所：統計學研究所 是否使用計算機：是
考試時間：100 分鐘 本科原始成績：100 分

1-5題每題20分，須附上該有之步驟。

1. 設隨機變數X_λ之機率密度函數(p.d.f.) 為

$$f(x) = 2\phi(x)\Phi(\lambda x),\ x \in R,$$

其中ϕ為$\mathcal{N}(0,1)$分佈之p.d.f., $\Phi(x) = \int_{-\infty}^{x} \phi(u)du$, $\lambda \in R$為一常數 。

(i) 試證$f(x)$確實為一p.d.f.,

(ii) 求$Y = X_\lambda^2$之p.d.f., 並決定此為那一常見分佈,

(iii) 求$\lambda \to \infty$時, X_λ之極限分佈,

(iv) 令$\Phi(x;\lambda) = P(X_\lambda \le x)$, $x \in R$ 。試證$\Phi(x;1) = (\Phi(x))^2$, $x \in R$ 。

2. 設$X_1, X_2, \cdots, X_n$, $n \ge 1$, 為i.i.d. $\mathcal{U}(0,1)$ 分佈之隨機變數, $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$表其順序統計量 。對$1 < k < n$, 試求

(i) 給定$X_{(k)} = a$, $0 < a < 1$, $X_{(1)}, \cdots, X_{(k-1)}$之條件分佈,

(ii) 給定$X_{(k)} = a$, $0 < a < 1$, $E(X_{(j)} - X_{(i)})$, 假設$k \le i < j \le n$ 。

3. 設隨機變數X滿足$E(X^2) < \infty$ 。令$[x]$表小於或等於x之最大整數 。則證明或否證：

(i) $E([X]) \le E(X)$是否必成立?

(ii) $\mathrm{Var}([X]) \le \mathrm{Var}(X)$是否必成立?

4. 設將r個球隨機地放進n個盒子中 。令A_i表第i個盒子為空盒之事件, N_n表總共之空盒數 。試證

(i) $P(A_i) = (1-1/n)^r, E(N_n) = n(1-1/n)^r, \mathrm{Var}(N_n) = n(n-1)((1-2/n)^r - (1-1/n)^{2r}) + n((1-1/n)^r - (1-1/n)^{2r})$;

(ii) 若r與n有關, 且$n \to \infty$時, $r/n \to c$, 則$n \to \infty$時, $E(N_n/n) \to e^{-c}$, 且N_n/n機率收斂(converges in probability) 至e^{-c} 。

5. 設隨機數列$\{X_n, n \ge 1\}$機率收斂至隨機變數X 。試證存在子一數列$\{n_k, k \ge 1\}$, 使得$k \to \infty$時, $\{X_{n_k}, k \ge 1\}$幾乎確實地收斂(converges almost surely)至X 。

國立高雄大學九十六學年度博士班招生考試試題

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1. Markov's Inequality is known as

$$P(|X| \geq \varepsilon) \leq E(|X|)/\varepsilon$$

for an arbitrary random variable X and $\varepsilon > 0$.

(a) For every fixed $\varepsilon > 1$, find a distribution for X with $E(X) = 0$ that gives equality in Markov's inequality. (10 points)

(b) Prove for an arbitrary random variable X,

$$P(|X| \geq \varepsilon) \leq \frac{E(\cosh(X)) - 1}{\cosh(\varepsilon) - 1}.$$

Here $\cosh(x) = (e^x + e^{-x})/2$. (10 points)

2. Let $X_1, \ldots, X_n$ be a sample from a population with the density

$$f(x) = \theta(\theta+1)x^{\theta-1}(1-x),$$

$0 < x < 1, \theta > 0$. Find a method of moments estimate of θ. (10 points)

3. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a sample from a $U(0, \theta)$ population, where $0 < \theta < \infty$ and θ is unknown. Then find the U.M.V.U. estimate of θ. (15 points)

4. Suppose that $X_1, \ldots, X_n$ is a sample from the pdf, $f(x|\theta) = \exp(-(x-\theta))$ $x > \theta$ and $\theta \in \Re$. Then construct the U.M.P. level α test for

$$H_0 : \theta \leq \theta_0 \text{ vs. } H_1 : \theta > \theta_0.$$

(15 points)

5. Let $X_1, \ldots, X_n$ be a sample from a $N(\mu, \sigma^2)$ population. Here σ^2 is assumed to be known. Suppose $z(p)$ denote the pth quantile of the standard normal distribution.

(a) Show $[\bar{X} - \sigma z(1-\alpha_1)/\sqrt{n}, \bar{X} + \sigma z(1-\alpha_2)/\sqrt{n}]$ is a level $(1-\alpha)$ confidence interval for μ, where $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $\alpha_1 + \alpha_2 = \alpha$. (10 points)

(b) From (a), the shortest level $(1-\alpha)$ interval is obtained by taking $\alpha_1 = \alpha_2 = \alpha/2$. (10 points)

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6. Let $X_1, \cdots, X_n$ be a sample from $N(\theta, \sigma^2)$ with σ^2 known. For a fixed number a, let $p = P(X_i > a)$.

   (a) Find the maximum likelihood estimate of p, $\hat{p}$. (10 pints)

   (b) Find the asymptotic distribution of $\sqrt{n}(\hat{p} - p)$. (10 points)