國立高雄大學九十七學年度博士班招生考試試題

科目：線性代數
考試時間：100 分鐘

系所：
應用數學系博士班
本科原始成績：100 分

是否使用計算機：是

Notations.

I_n : the identity matrix of size n.
$M_{n\times m}(\mathbb{R})$: set of $n\times m$ real matrices.

1 (24) Determine "true" or "false" for the following statements. Briefly sketch your proof when the answer is "true", or give a counterexample when the answer is "false".
a. If $A^2 = A$ then the eigenvalues of A are either 0 or 1.
b. Let $A, B \in M_{n\times n}(\mathbb{R})$. If $AB = 0$ then rank(A)+rank$(B) \leq n$.
c. If $A, B \in M_{n\times n}(\mathbb{R})$ then AB and BA have the same minimal polynomial.
d. If $A, B \in M_{n\times n}(\mathbb{R})$ then the determinant of $AB - BA$ is zero.

2 Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$.
a. (6) Find the projection of $\mathbf{b}$ onto the column space of A.
b. (6) Let $\mathbf{b}_A$ be the projection of $\mathbf{b}$ onto the column space of A. Find solution $\mathbf{s}$ to $A\mathbf{x} = \mathbf{b}_A$ such that $\|\mathbf{s}\| \leq \|\mathbf{u}\|$ for all other solutions $\mathbf{u}$.

3 (10) Let $A \in M_{n\times m}(\mathbb{R})$, $B \in M_{m\times n}(\mathbb{R})$. Prove that if λ is an eigenvalue of AB and $\lambda \neq 0$ then λ is an eigenvalue of BA.

4 (10) Let $A \in M_{n\times n}(\mathbb{R})$ and $r \in \mathbb{R}$. Suppose $A - rI_n = QR$ where Q is orthogonal and R is upper-triangular. Let $B = RQ + rI_n$. Show that A and B have the same eigenvalues.

5 Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $A = I_n - \mathbf{x}\mathbf{y}^T$.
a. (5) Let $\mathbf{x}^T\mathbf{y} \neq 1$. Show that

$$A^{-1} = I_n - \frac{1}{\mathbf{x}^T\mathbf{y} - 1}\mathbf{x}\mathbf{y}^T.$$

b. (8) Let $\mathbf{x}^T\mathbf{y} \neq 0$. Find the eigenvalues and corresponding to eigenspaces of A.
c. (5) Show that det(A)=$1 - \mathbf{x}^T\mathbf{y}$.
d. (8) Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors. Show that if $\mathbf{x}^T\mathbf{y} = 0$ then A is not diagonalizable.

6 Let $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$. Define $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ such that $T(X) = XA - AX$.
a. (5) Show that T is a linear transformation.
b. (10) Find the eigenvalues and eigenvectors of T.
c. (3) Find the minimal polynomial of T.

國立高雄大學九十七學年度博士班招生考試試題

科目：高等微積分
考試時間：100 分鐘

系所：
應用數學系博士班
本科原始成績：100 分

是否使用計算機：是

1. (15%) Suppose that O is open set on $\mathbb{R}$. Let $B = \left\{(x,0) \in \mathbb{R}^2 \mid x \in O\right\}$. Does B open on $\mathbb{R}^2$? Please, explain why true or false.

2. (15%) Let $f : [a,b] \to \mathbb{R}$ be a smooth function, let S be the surface obtained by revolving the curve $y = f(x)$, $a \le x \le b$, about the x axis. Prove that the surface area of S is
$$A(S) = 2\pi \int_a^b |f(x)| \sqrt{1 + |f'(x)|^2} dx.$$

3. (15%) Find an example of a sequence f_k that converges uniformly to zero on $[0,\infty)$, where each $\int_0^\infty f_k(x)\,dx$ exists, but
$$\int_0^\infty f_k(x)\,dx \to \infty \text{ as } k \to \infty.$$

4. (15%) Prove that there is no continuous function taking $[0,1]$ onto $(0,1)$.

5. (15%) Find the value of surface integral: $\int\int_S F \cdot \overrightarrow{n} d\sigma$, where $F(x,y,z) = (xy, x-y, z)$, S is the planar region $x+y+z=1, (x,y) \in [0,1] \times [0,1]$, and $\overrightarrow{n}$ is the downward-pointing normal.

6. (10%) Find the value of line integral: $\int_C g ds$ where $g(x,y) = 2x+y, C = \phi(I), \phi(t) = (\cos t, \sin t)$ and $I = [0, \pi/2]$

7. (15%) Prove that if
(i) $f_n(x), g(x)$ are continuous,
(ii) $|f_n(x)| \le g(x)$, $n = 1, 2, \ldots$, $0 \le x < \infty$,
(iii) $f_n(x) \to f(x)$ uniformly, $0 \le x \le R$, for any $R < \infty$, and
(iv) $\int_0^\infty g(x)\,dx < \infty$,
then $\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx \to \int_0^\infty f(x)\,dx$.